The classical Lie algebras

We have already seen $\mathfrak{sl}_n$: the Lie algebra of

$$\{ X \in \mathrm{Mat}_{n\times n}(K) \mid \mathrm{Tr}(X) = 0 \}$$

In this lecture, we will learn about

- $\mathfrak{so}_n$ the orthogonal Lie algebra,
- $\mathfrak{sp}_{2n}$ the symplectic Lie algebra.

The symplectic Lie algebra

Let V be a $2n$-dimensional vector space over $K = \bar{K}$ (char $K = 0$)

and let $Q : V \times V \to K$ be a nondegenerate skew-symmetric bilinear form on V. (Alternative: $Q : \Lambda^2 V \to K$)

The symplectic group $Sp(V, Q)$ is

$$\{ A \in \mathrm{Aut}(V) \mid Q(Av, Aw) = Q(v, w) \}$$

If $K = \mathbb{C}$, this is a Lie group.

We can rewrite the condition $Q(Av, Aw) = Q(v,w)$ as

$$Q(Av, w) = Q(v, A^{-1}w)$$

Differentiating: $\quad \mathrm{Aut} \rightsquigarrow \mathrm{End}$ and $A^{-1} \rightsquigarrow -A$.

This leads us to the symplectic Lie algebra $\mathfrak{sp}(V,Q)$:

$$\{ A \in \mathrm{End}(V) \mid Q(Av, w) = -Q(v, Aw) \}$$

Of course we can also write the condition as

$$Q(Av, w) + Q(v, Aw) = 0 .$$

Explicit example / choosing a basis

One can choose a basis of V, $e_1, \ldots, e_{2n}$ such that

$$Q(e_i, e_j) = \begin{cases} 1 & \text{if } j = i+n \\ -1 & \text{if } i = j+n \\ 0 & \text{otherwise} \end{cases}$$

In terms of this basis we can find a matrix M so that

$$Q(v, w) = {}^t v \cdot M \cdot w$$

Explicitly

$$M = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in Mat_{2n \times 2n}(K)$$

where I_n is the identity matrix.

Under these choices, $\mathfrak{sp}(V, Q)$ is identified with

$$\mathfrak{sp}_{2n} = \mathfrak{sp}_{2n}(K) = \left\{ X \in \mathrm{Mat}_{2n\times 2n}(K) \;\middle|\; {}^tX \cdot M + M \cdot X = 0 \right\}$$

If we write $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then we can compute

$${}^tX \cdot M = \begin{pmatrix} -{}^tC & {}^tA \\ -{}^tD & {}^tB \end{pmatrix} \qquad M \cdot X = \begin{pmatrix} C & D \\ -A & -B \end{pmatrix}$$

so the conditions are : ${}^tA = -D$, and B, C symmetric.

<u>Exercise</u> Check these computations.

The orthogonal Lie algebra

Let V be a m-dimensional vector space over $K = \overline{K}$ $(\operatorname{char} K = 0)$

and let $Q : V \times V \to K$ be a nondegenerate symmetric

bilinear form on V. $\left(\text{Alternative: } Q : \operatorname{Sym}^2(V) \to K \right)$

The special orthogonal group $SO_m(V, Q)$ is

$$\{ A \in \operatorname{Aut}(V) \mid Q(Av, Aw) = Q(v,w) \text{ and } \det(A) = 1 \}$$

If $K = \mathbb{C}$, this is a Lie group.

We can rewrite the condition $Q(Av, Aw) = Q(v,w)$ as

$$Q(Av, w) = Q(v, A^{-1}w)$$

Differentiating: $\mathrm{Aut} \rightsquigarrow \mathrm{End}$ and $A^{-1} \rightsquigarrow -A$.

This leads us to the orthogonal Lie algebra $\mathfrak{so}(V, Q)$:

$$\{ A \in \mathrm{End}(V) \mid Q(Av, w) = -Q(v, Aw) \}$$

Of course we can also write the condition as

$$Q(Av, w) + Q(v, Aw) = 0 .$$

Explicit example / choosing a basis

We distinguish $m = 2n$ and $m = 2n+1$. Suppose first that $m = 2n$.

Then we can find a basis $e_1, \dots, e_{2n}$ such that

$$Q(e_i, e_j) = \begin{cases} 1 & \text{if } j = i \pm n \\ 0 & \text{otherwise} \end{cases}$$

In terms of this basis we can find a matrix M so that

$$Q(v, w) = {}^t v \cdot M \cdot w.$$

Namely:
$$M = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \in Mat_{2n \times 2n}(K).$$

Under these choices, $\mathfrak{so}(V,Q)$ is identified with

$$\mathfrak{so}_{2n} = \mathfrak{so}_{2n}(K) = \left\{ X \in \mathrm{Mat}_{2n\times 2n}(K) \;\middle|\; {}^tX M + M X = 0 \right\}$$

If we write $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then we can compute

$${}^tX \cdot M = \begin{pmatrix} {}^tC & {}^tA \\ {}^tD & {}^tB \end{pmatrix} \qquad M \cdot X = \begin{pmatrix} C & D \\ A & B \end{pmatrix}$$

so the conditions are: ${}^tA = -D$, and B, C skew-symmetric.

<u>Exercise</u> Check these computations.

Now consider the case $m = 2n+1$.

Then we can find a basis $e_1, \dots, e_{2n+1}$ such that

$$Q(e_i, e_j) = \begin{cases} 1 & \text{if } (i,j \leq 2n \text{ and } i = j \pm n) \text{ or } (i = j = 2n+1) \\ 0 & \text{otherwise} \end{cases}$$

In terms of this basis we can find a matrix M so that

$$Q(v, w) = {}^t v \cdot M \cdot w.$$

Namely:

$$M = \begin{pmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Mat}_{m \times m}(K) .$$

Under these choices, $\mathfrak{so}(V, Q)$ is identified with

$$\mathfrak{so}_{2n+1} = \mathfrak{so}_{2n+1}(K) = \left\{ X \in \mathrm{Mat}_{m \times m}(K) \,\middle|\, {}^tX M + M X = 0 \right\}$$

If we write $X = \begin{pmatrix} A & B & E \\ C & D & F \\ G & H & J \end{pmatrix}$, then we can compute

that ${}^tXM + MX = 0$ is equivalent to the conditions

$A = -{}^tD$, B, C skew-symmetric, and $E = -{}^tH$, $F = -{}^tG$, $J = 0$.

<u>Exercise</u> Check these computations.