Casimir elements

(All Lie algebras and vector spaces are over a given field K.)

Let $\mathfrak{g}$ be a Lie algebra, and V a representation of $\mathfrak{g}$.

An element $v \in V$ is invariant if $X(v) = 0$ for all $X \in \mathfrak{g}$.

Examples Let V, W be representations of $\mathfrak{g}$.

$\mathfrak{g}$ acts on $\mathrm{Hom}_K(V,W)$ via $X(f)(v) = X(f(v)) - f(X(v))$.

Then $f \in \mathrm{Hom}_K(V,W)$ is invariant $\iff$ f is a morphism of $\mathfrak{g}$-representations

Examples (cntd.)

Let $e_1, \dots, e_n$ be a basis of V and

$e_1^*, \dots, e_n^*$ the dual basis of V^*.

We have $\mathrm{Hom}_k(V, V) \cong V^* \otimes V$

$$\mathbb{1} = \mathrm{id} \longleftrightarrow \sum_{i=1}^{n} e_i^* \otimes e_i$$

Hence $\sum_{i=1}^{n} e_i^* \otimes e_i$ is invariant.

Examples (cntd.) Let β be a bilinear form on V.

We can view β as element of $\mathrm{Hom}_K(V, V^*) \cong V^* \otimes V^*$.

Suppose that β is nondegenerate and invariant,

so that β induces an isomorphism $V \xrightarrow{\sim} V^*$ of $\mathfrak{g}$-reps.

Let $e_1, \dots, e_n$ and $e_1', \dots, e_n'$ be two bases of V such that $\beta(e_i, e_j') = \delta_{ij}$

Then
$$\begin{array}{ccc} \mathrm{Hom}_K(V, V^*) \cong V^* \otimes V^* \cong V \otimes V & & \\ \beta \longleftrightarrow \sum_{i=1}^{n} e_i \otimes e_i' & & \text{is invariant.} \end{array}$$

What does it mean that β is invariant?

$$X(\beta)(v,w) = X(\beta(v,w)) - \beta(X(v),w) - \beta(v,X(w))$$

$$\overset{\shortparallel}{0} \qquad\qquad \overset{\shortparallel}{0}$$

Now suppose that $V = \mathfrak{g}$. Then, for $X, Y, Z \in \mathfrak{g}$

$$\beta([X,Y],Z) = -\beta(Y,[X,Z])$$

$$\beta([Y,X],Z) = \beta(Y,[X,Z])$$

Example: Killing form!

Let B_V be the bilinear form attached to the representation

$\rho : \mathfrak{g} \to \mathfrak{gl}(V)$

$$B_V : \mathfrak{g} \times \mathfrak{g} \longrightarrow K$$
$$(X, Y) \longmapsto \mathrm{Tr}(\rho(X) \circ \rho(Y)).$$

Let $e_1, \ldots, e_n$ and $e_1', \ldots, e_n'$ be two bases of $\mathfrak{g}$ such that $B_V(e_i, e_j') = \delta_{ij}$.

Let W be any representation of $\mathfrak{g}$.

Then $C = C_V : W \to W$

$$w \longmapsto \sum_{i=1}^{n} e_i(e_i'(w))$$

is the Casimir operator of V.

(It is an endomorphism of W, for every W.)

Lemma The Casimir operator does not depend on the choice of basis, and is invariant.

Proof It is the image of the invariant element

$$c = \sum_{i=1}^{n} e_i \otimes e^i \in \mathfrak{g} \otimes \mathfrak{g}$$

which does not depend on the choice of basis.

$\dim(\mathfrak{g})$

<u>Lemma</u> Assume that $\cancel{\dim(V)} = n$.

- $\Box$ $\mathrm{Tr}(C_V : V \to V) = n$

- $\Box$ If V is simple and $\mathrm{char}(K) \nmid n$, then C_V is an automorphism of V.

<u>Proof</u> $\mathrm{Tr}(C_V) = \sum_{i=1}^{n} \mathrm{Tr}(e_i \circ e_i^{\vee}) = \sum_{i=1}^{n} B_V(e_i, e_i^{\vee}) = n$.

If $\mathrm{char}(K) \nmid n$, then $C_V \neq 0$. If V is simple then $\mathrm{Ker}(C_V) = 0$ hence C_V is an automorphism.