Cartan subalgebra

($K = \bar{K}$ a field of char. 0)

Let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra over K.

A Cartan subalgebra of $\mathfrak{g}$ is an abelian subalgebra of $\mathfrak{g}$ consisting of semisimple elements and not contained in any larger such subalgebra.

Goal: existence of Cartan subalgebras.

For $H \in \mathfrak{g}$, let

$$\mathfrak{c}(H) = \{ X \in \mathfrak{g} \mid [H, X] = 0 \}$$

be the centraliser of H.

For a subalgebra $\mathfrak{h} \subset \mathfrak{g}$, the normaliser is

$$\{ X \in \mathfrak{g} \mid [\mathfrak{h}, X] \subset \mathfrak{h} \}.$$

The plan:

Alternative definition

Let $\mathfrak{g}$ be Lie algebra. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a **Cartan subalgebra** if $\mathfrak{h}$ is **nilpotent** and equal to its own **normaliser.**

- ☐ Show that this definition implies the properties on page 1.
- ☐ Show that Cartan subalgebras in the above sense **exist.**

Let H be an element of $\mathfrak{g}$.

We can decompose $\mathfrak{g}$ for the adjoint action of H:

$$\mathfrak{g} = \bigoplus_{\lambda} \mathfrak{g}_\lambda(H)$$

where $\mathfrak{g}_\lambda(H) = \{X \in \mathfrak{g} \mid (\mathrm{ad}\,H - \lambda I)^k X = 0 \text{ for some } k\}$

<u>Exercise</u> Show $[\mathfrak{g}_\lambda(H), \mathfrak{g}_\mu(H)] \subset \mathfrak{g}_{\lambda+\mu}(H)$.

In particular $\mathfrak{g}_0(H)$ is a subalgebra of $\mathfrak{g}$.

An element $X \in \mathfrak{g}$ is called

- generic if $\dim \mathfrak{g}_0(X) \leq \dim \mathfrak{g}_0(Y)$ for all $Y \in \mathfrak{g}$
- regular if $\dim \mathfrak{c}(X) \leq \dim \mathfrak{c}(Y)$ for all $Y \in \mathfrak{g}$

The rank of $\mathfrak{g}$ is the smallest dimension of $\mathfrak{c}(H)$ where H ranges over all elements of $\mathfrak{g}$.

In particular: H is regular $\iff$ $\dim \mathfrak{c}(H)$ is equal to $\operatorname{rank}(\mathfrak{g})$.

<u>Subgoal:</u> If H is generic,

then $\mathfrak{g}_0(H)$ is a Cartan subalgebra.

<u>Observation</u> If $X \in \mathfrak{g}$ and $[X, Y] \in \mathfrak{g}_0(H)$ for all $Y \in \mathfrak{g}_0(H)$,

then $X \in \mathfrak{g}_0(H)$.

Proof: Exercise. (Hint: note that $[H, X] \in \mathfrak{g}_0(H)$.)

<u>Corollary</u> $\mathfrak{g}_0(H)$ is its own normaliser.

Claim If H is generic, then $\mathfrak{g}_0(H)$ is nilpotent.

Let $\mathfrak{n} \subset \mathfrak{g}_0(H)$ be a nilpotent subalgebra of maximal dimension that contains x. We get a decomposition

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{n}^*} \mathfrak{g}_\lambda(\mathfrak{n})$$

where

$$\mathfrak{g}_\lambda(\mathfrak{n}) = \left\{ X \in \mathfrak{g} \;\middle|\; \begin{array}{l} (ad(n) - \lambda(n) I)^k X = 0 \text{ for all } n \in \mathfrak{n} \\ \text{and } k \text{ large enough} \end{array} \right\}$$

Note that $\mathfrak{g}_\lambda(\mathfrak{n}) \subset \mathfrak{g}_{\lambda(H)}(H)$, and in particular $\mathfrak{g}_0(\mathfrak{n}) \subset \mathfrak{g}_0(H)$.

Let $\lambda_1, \ldots, \lambda_r \in \mathfrak{h}^*$ be the nonzero "weights" such that $\mathfrak{g}_{\lambda_i}(\mathfrak{h}) \neq 0$.

Suppose that $\mathfrak{g}_0(\mathfrak{h}) \subset \mathfrak{g}_0(H)$ is not an equality.

Then there exists a $y \in \mathfrak{h}$ such that

$$\lambda_1(y) \neq 0, \ldots, \lambda_r(y) \neq 0$$

because the condition $\{y \mid \lambda_i(y) = 0\}$ is a hyperplane in $\mathfrak{h}$

and the complement of finitely many hyperplanes is nonempty.

But then $\mathfrak{g}_0(y) \subset \mathfrak{g}_0(\mathfrak{h}) \subsetneq \mathfrak{g}_0(H)$. But H is generic $\lightning$

Hence $\mathfrak{g}_0(\mathfrak{h}) = \mathfrak{g}_0(H)$.

Now we want to argue that $\mathfrak{n} = \mathfrak{g}_0(H)$ using the assumption that $\mathfrak{n}$ is maximal.

Suppose that $\mathfrak{n} \subsetneq \mathfrak{g}_0(H)$.

Since $\mathfrak{n}$ acts on $\mathfrak{g}_0(H)/\mathfrak{n}$ via nilpotent endomorphisms there exists an element $z \in \mathfrak{g}_0(H)$ with $z \notin \mathfrak{n}$ and $[z, \mathfrak{n}] \subset \mathfrak{n}$.

Hence $\mathfrak{m} = K \cdot z + \mathfrak{n}$ is a solvable Lie algebra.

We continue with a similar argument as before:

Let $\mu_0, \dots, \mu_t \in \mathfrak{m}^*$ be the functionals through which $\mathfrak{m}$ acts on irreducible subquotients of $\mathfrak{g}_0(H)$.

We may assume $\mu_0 = 0$ since $[z, z] = 0$.

By the same argument with hyperplanes, there exists a $y \in \mathfrak{m}$ such that $\mu_1(y) \neq 0, \dots, \mu_t(y) \neq 0$.

Once again $\mathfrak{g}_0(y) \subset \mathfrak{g}_0(H)$, and this must be an equality since H is generic. But then $\mathfrak{m}$ does not have nonzero eigenvalues on $\mathfrak{g}_0(H)$. In other words $t = 0$ and $\mathfrak{m}$ is nilpotent. $\lightning$

Hence $\mathfrak{n} = \mathfrak{g}_0(H)$.

Conclusion $\mathfrak{g}_0(H)$ is a Cartan subalgebra.

Now show: If $\mathfrak{g}$ is semisimple, then

- $\mathfrak{g}_0(H)$ is abelian
- $\mathfrak{g}_0(H)$ consists of semisimple elements of $\mathfrak{g}$.

<u>Claim</u> $\mathfrak{g}_0(H)$ is perpendicular to $\mathfrak{g}_\lambda(H)$ (for $\lambda \neq 0$) with respect to the Killing form B.

<u>Proof.</u> Suppose that $Y \in \mathfrak{g}_\lambda(H)$ and $X \in \mathfrak{g}_0(H)$.

By the exercise on page 4, ad Y maps $\mathfrak{g}_\mu(H)$ to $\mathfrak{g}_{\lambda+\mu}(H)$, and so does $\mathrm{ad}\, Y \circ \mathrm{ad}\, X$.

In particular $\mathrm{ad}\, Y \circ \mathrm{ad}\, X$ will have zeroes on the diagonal, when written as matrix w.r.t. a basis compatible with the decomposition.

This means that $B(Y, X) = \mathrm{Tr}(\mathrm{ad}\, Y \circ \mathrm{ad}\, X) = 0$.

Since $\mathfrak{g}$ is semisimple, B is nondegenerate

and by the computation above $B|_H$ is also nondegenerate.

By Lie's theorem, there is a basis of $\mathfrak{g}$ such that

$\mathrm{ad}\, X$ is an upper-triangular matrix for all $X \in \mathfrak{g}_0(H)$.

This means that for $X_1, \ldots, X_n \in \mathfrak{g}_0(H)$

$$\mathrm{Tr}(\mathrm{ad}\, X_1 \circ \ldots \circ \mathrm{ad}\, X_n) = \mathrm{Tr}(\mathrm{ad}\, X_{\sigma(1)} \circ \ldots \circ \mathrm{ad}\, X_{\sigma(n)})$$

for all permutations $\sigma : \{1, \ldots, n\} \xrightarrow{\sim} \{1, \ldots, n\}$.

Now let $X, Y, Z \in \mathfrak{g}_0(H)$ be arbitrary.

Then
$$B([X,Y], Z) = Tr(ad[X,Y] \circ ad\, Z)$$
$$= Tr(ad\, X \circ ad\, Y \circ ad\, Z) - Tr(ad\, Y \circ ad\, X \circ ad\, Z)$$
$$= 0$$

But Z is arbitrary and B is nondegenerate.

Hence $[X,Y] = 0$, which means that $\mathfrak{g}_0(H)$ is abelian.

Finally, we show that $\mathfrak{g}_0(H)$ consists of semisimple elements of $\mathfrak{g}$.

Fix $X \in \mathfrak{g}_0(H)$. Let $X = X_s + X_n$ be the Jordan decomposition in $\mathfrak{g}$.

We will show $X_n = 0$.

Using the same basis as above, $\mathrm{ad}(X_n)$ is

strictly upper-triangular.

Hence $B(X_n, Y) = \mathrm{Tr}(\mathrm{ad}\, X_n \circ \mathrm{ad}\, Y) = 0$ for all $Y \in \mathfrak{g}_0(H)$.

Since B is nondegenerate, we conclude $X_n = 0$, hence $X = X_s$. $\square$