Cartan's criterion Let $\mathfrak{g}$ be a Lie algebra.

(i) $\mathfrak{g}$ is solvable if and only if $B(\mathfrak{g}, \mathcal{D}\mathfrak{g}) = 0$.

(ii) If $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(V)$ and $B_V(X, Y) = 0$ for all $X \in \mathcal{D}\mathfrak{g}$, $Y \in \mathfrak{g}$, then $\mathfrak{g}$ is solvable.

(iii) $\mathfrak{g}$ is semisimple if and only if its Killing form B is nondegenerate.

Recap: Jordan–Chevalley decomposition

Let X be an endomorphism of a vector space V / perfect field

Then there exists a decomposition $X = X_s + X_n$ with

- X_s semisimple (= diagonalizable)
- X_n nilpotent
- X_s and X_n commute

Such a decomposition is unique, and X_s and X_n are polynomial expressions in X without constant coefficients.

Lemma Let X be an endomorphism of V, with JC-decomposition

$$X = X_s + X_n$$

Consider $\mathrm{ad}(X)$, an endomorphism of $\mathfrak{gl}(V)$.

Then $\mathrm{ad}(X)_s = \mathrm{ad}(X_s)$ and $\mathrm{ad}(X)_n = \mathrm{ad}(X_n)$.

Proof Clearly $\mathrm{ad}(X) = \mathrm{ad}(X_s) + \mathrm{ad}(X_n)$,

and $\mathrm{ad}(X_n)$ is nilpotent. Also $[\mathrm{ad}(X_s), \mathrm{ad}(X_n)] = 0$

Choose a basis $v_1, \ldots, v_k$ of V on which X_s is diagonal.

Choose a basis $v_1, \dots, v_k$ of V on which X_s is diagonal.

In other words: $X_s(v_i) = \lambda_i \cdot v_i$, for eigenvalues λ_i.

This basis allows us to identify $\mathfrak{gl}(V) = \mathrm{Mat}_{k\times k}(K)$.

On the standard basis $(E_{ij})_{ij}$ of $\mathrm{Mat}_{k\times k}(K)$

we can compute that $\mathrm{ad}(X_s)(E_{ij}) = (\lambda_i - \lambda_j)\cdot E_{ij}$.

Hence $\mathrm{ad}(X_s)$ is diagonal. $\square$

Lemma Suppose that A and B are two subspaces of $\mathfrak{gl}(V)$, and $B \subset A$. Let T be the subspace to $t \in \mathfrak{gl}(V)$ for which $[t, A] \subset B$. If $z \in T$, and $\mathrm{Tr}(z \circ u) = 0$ for all $u \in T$, then z is nilpotent.

Proof Let $z = s + n$ be the JC-decomposition, and let $v_1, \ldots, v_k$ be a basis of V on which s is diagonal. Let $\lambda_i \in K$ be the corresponding eigenvalues, so that $s(v_i) = \lambda_i \cdot v_i$.

Let λ_i be the corresponding eigenvalue, so that $s(v_i) = \lambda_i$.

We want to show $\lambda_i = 0$, so that $z = n$ is nilpotent.

Consider the $\mathbb{Q}$-sub-vector space $\Lambda \subset K$ generated by the λ_i.

Let $f : \Lambda \rightsquigarrow \mathbb{Q}$ be a $\mathbb{Q}$-linear function. We will show $f = 0$.

Define $t \in \mathfrak{gl}(V)$ via $t(v_i) = f(\lambda_i) v_i$.

On the standard basis $(E_{ij})_{ij}$ of $\mathfrak{gl}(V)$ induced by $(v_i)_i$:

$$\mathrm{ad}(s) E_{ij} = (\lambda_i - \lambda_j) E_{ij}$$
$$\mathrm{ad}(t) E_{ij} = (f(\lambda_i) - f(\lambda_j)) E_{ij}$$

$$\mathrm{ad}(s)\, E_{ij} = (\lambda_i - \lambda_j)\, E_{ij}$$

$$\mathrm{ad}(t)\, E_{ij} = (f(\lambda_i) - f(\lambda_j))\, E_{ij}$$

By Lagrange interpolation, there is a polynomial $P \in K[X]$ such that $P(\lambda_i - \lambda_j) = f(\lambda_i) - f(\lambda_j)$.

Well-defined, because $\lambda_i - \lambda_j = \lambda_l - \lambda_m \Rightarrow f(\lambda_i) - f(\lambda_j) = f(\lambda_l) - f(\lambda_m)$

Note that the constant coefficient of P is 0:

$$P(0) = P(\lambda_i - \lambda_i) = f(\lambda_i) - f(\lambda_i) = 0$$

$$\operatorname{ad}(s) E_{ij} = (\lambda_i - \lambda_j) E_{ij}$$

$$\operatorname{ad}(t) E_{ij} = (f(\lambda_i) - f(\lambda_j)) E_{ij}$$

$$P(\lambda_i - \lambda_j) = f(\lambda_i) - f(\lambda_j)$$

By construction: $\operatorname{ad}(t) = P(\operatorname{ad}(s))$

From the first lemma: $\operatorname{ad}(s) = \operatorname{ad}(z)_s$ is a polynomial expression in $\operatorname{ad}(z)$ without constant coefficient.

By assumption $(\operatorname{ad} z)(A) \subset B$, and hence

$(\operatorname{ad} t)(A) \subset B$, so $t \in T$.

Recall the hypothesis on z and the goal from the lemma:

> If $z \in T$, and $\mathrm{Tr}(z \circ u) = 0$ for all $u \in T$,
> then z is nilpotent.

We just showed $t \in T$, and therefore

$$0 = \mathrm{Tr}(zt) = \sum \lambda_i f(\lambda_i).$$

Apply f to both sides:

$$0 = f(0) = f\left(\sum \lambda_i f(\lambda_i)\right) = \sum f(\lambda_i)^2.$$

But $f(\lambda_i) \in \mathbb{Q}$, hence $f(\lambda_i) = 0$ and therefore $\lambda_i = 0$. $\square$

Proof of Cartan's criterion

If $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(V)$ and $B_V(X,Y)=0$ for all $X \in D\mathfrak{g}$, $Y \in \mathfrak{g}$, then $\mathfrak{g}$ is solvable.

Let T be the set of $t \in \mathfrak{gl}(V)$ for which $[t,\mathfrak{g}] \subset D\mathfrak{g}$.

For $t \in T$, $X,Y \in \mathfrak{g}$, we have $[t,X] \subset D\mathfrak{g}$, hence

$$\mathrm{Tr}(t \circ [X,Y]) = B_V(t,[X,Y]) = B_V([t,X],Y) = 0$$

By linearity $\mathrm{Tr}(t \circ u) = 0$ for all $u \in D\mathfrak{g}$.
By the previous lemma: all $u \in D\mathfrak{g}$ act nilpotent.

By linearity $\mathrm{Tr}(tu) = 0$ for all $u \in \mathcal{D}\mathfrak{g}$.

By the previous lemma: all $u \in \mathcal{D}\mathfrak{g}$ act nilpotent.

Hence $\mathcal{D}\mathfrak{g}$ is a nilpotent Lie algebra. (Engel's thm)

This in turn implies that $\mathfrak{g}$ is solvable. $\square$