Basic definitions

- ☐ Representations
  - sub
  - quot
- ☐ Subalgebra
- ☐ Ideal

Working towards

Two theorems that give a "rough" description of the structure of Lie algebras:

- ☐ Engel's theorem
- ☐ Lie's theorem

Recap

Let R be a ring.

A module over R consists of an abelian group M and a scalar multiplication

$$R \times M \to M$$

satisfying some axioms

Alternative:

An abelian group M and a ring homomorphism

$$R \longrightarrow \mathrm{End}(M)$$

Notation

We write $\mathfrak{gl}(V)$ for the Lie algebra

$$\mathrm{End}(V)$$

with the commutator bracket .

Definition

Let $\mathfrak{g}$ be a Lie algebra (over K).

A **representation** of $\mathfrak{g}$ consists of

- a K-vector space V
- a **morphism** of Lie algebras

$$\mathfrak{g} \xrightarrow{\rho} \mathfrak{gl}(V)$$

Notation: $X(v) = \rho(X)(v)$ for $X \in \mathfrak{g}$, $v \in V$

Concretely

Let V be a $\mathfrak{g}$ representation: $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$.

Then $X(v) \in V$ for all $X \in \mathfrak{g}$, $v \in V$.

$$[X, Y](v) = X(Y(v)) - Y(X(v))$$

Definition

A representation V of $\mathfrak{g}$

is called faithful

if $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$

is injective

Adjoint representation

Check that the map

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$$

$$X \longmapsto [X, -]$$

is a Lie algebra homomorphism.

It is the adjoint representation of $\mathfrak{g}$.

Subrepresentations

Let V be a representation of $\mathfrak{g}$.

$W \subset V$ is a subrepresentation if it is a sub-vector space, and $X(w) \in W$, for all $X \in \mathfrak{g}$, $w \in W$

Quotient representations

Let $W \subset V$ be a subrepresentation.

The quotient representation on V/W

is defined via:

$$X(\bar{v}) = \overline{X(v)}$$

Check: $\square$ well-defined

$\square$ satisfies the condition of a representation

Ideals and subalgebras

An ideal $\mathfrak{h} \subset \mathfrak{g}$ of a Lie algebra is a subrepresentation of the adjoint representation:

$$[X, Y] \in \mathfrak{h} \quad \text{for all } X \in \mathfrak{g}, \ Y \in \mathfrak{h}.$$

A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of a Lie algebra is a sub-vector space satisfying:

$$[X, Y] \in \mathfrak{h} \quad \text{for all } X, Y \in \mathfrak{h}.$$

Note: every ideal is a subalgebra.

Exercises

(i) Let $f : \mathfrak{g} \to \mathfrak{h}$ be a morphism of Lie algebras

Show that $\ker(f)$ is an ideal of $\mathfrak{g}$.

(ii) Let $\mathfrak{g}$ be a Lie algebra and

$\mathfrak{h} \subset \mathfrak{g}$ an ideal.

Show that $\mathfrak{g}/\mathfrak{h}$ is a Lie algebra

in a natural way.

It has the universal property of quotients.