Jordan decomposition in a semisimple Lie algebra

Goal: preservation of the Jordan decomposition

We used this when we studied representations

of $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$.

Now it is time to prove it.

Fix an alg. closed field K of char. 0.

All Lie algebras and vector spaces will be over K.

Theorem Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra.

For every $X \in \mathfrak{g}$, there is a unique decomposition

$$X = s + n,$$

with $\mathrm{ad}(s)$ semisimple, $\mathrm{ad}(n)$ nilpotent, and $[s,n]=0$.

This is the absolute Jordan decomposition of X.

Theorem (cntd)

- If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a finite-dimensional representation, then $\rho(s) + \rho(n)$ is the usual JD of $\rho(X)$.

- If $\phi : \mathfrak{g} \to \mathfrak{g}'$ is a morphism of semisimple Lie algebras, then $\phi(X) = \phi(s) + \phi(n)$ is the absolute Jordan decomposition of $\phi(X)$.

Lemma Let V be a finite-dimensional vector space, and let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a semisimple Lie algebra.

For $X \in \mathfrak{g}$, let $X = s + n \in \mathrm{End}(V)$ be the Jordan decomposition.

Then $s \in \mathfrak{g}$ and $n \in \mathfrak{g}$.

Proof Idea: write $\mathfrak{g}$ as intersection of subalgebras of $\mathfrak{gl}(V)$ and show that s and n lie in those subalgebras.

For every subrepresentation $W \subset V$, define

$$S_W = \left\{ Y \in \mathfrak{gl}(V) \,\middle|\, Y(W) \subset W \text{ and } \operatorname{Tr}(Y|_W) = 0 \right\}$$

Note that $\mathfrak{g} = \mathcal{D}\mathfrak{g}$ since $\mathfrak{g}$ is semisimple.

Hence $X \in \mathfrak{g}$ is a commutator, and therefore $\operatorname{Tr}(X|_W) = 0$.

In particular $\mathfrak{g} \subset S_W$.

But then also $\operatorname{Tr}(s) = 0 = \operatorname{Tr}(n)$.

By "functoriality of the Jordan decomposition" we get $s, n \in S_W$.

Next up, we consider

$$\mathfrak{n} = \{ Y \in \mathfrak{gl}(V) \mid [Y, \mathfrak{g}] \subset \mathfrak{g} \}$$

Certainly $\mathfrak{g} \subset \mathfrak{n}$, and again, by "functoriality of JD"

we also find $s \in \mathfrak{n}$ and $n \in \mathfrak{n}$.

Now consider $\mathfrak{g}' = \mathfrak{n} \cap \bigcap_{\substack{W \subset V \\ W \text{ irred}}} s_W$.

We know $\mathfrak{g} \subset \mathfrak{g}'$ is an ideal, since $\mathfrak{g} \subset \mathfrak{n}$, and we know

$s \in \mathfrak{g}'$ and $n \in \mathfrak{g}'$. Claim: $\mathfrak{g} = \mathfrak{g}'$.

Now consider $\mathfrak{g}' = \mathfrak{n} \cap \bigcap_{\substack{W \subset V \\ \text{subrep.}}} \mathfrak{s}_W$. Claim: $\mathfrak{g} = \mathfrak{g}'$.

By a prep. lemma, we find an ideal $I \subset \mathfrak{g}'$ with $\mathfrak{g}' = \mathfrak{g} \oplus I$.

Let W be an irreducible summand of V. Pick $Y \in I$.

We find $X(Y(v)) = Y(X(v))$ for all $X \in \mathfrak{g}$, and $v \in W$.

Hence Y acts as $\mathfrak{g}$-invariant endomorphism of W.

By Schur's lemma $Y(v) = \lambda_W \cdot v$, for some $\lambda_W \in K$.

Since $\mathrm{Tr}(Y|_W) = 0$, we find $\lambda_W = 0$.

By complete reducibility $V = \bigoplus W_i$, W_i irreducible.

We found $\lambda_{W_i} = 0$ for all i, hence $Y = 0$.

This means $I = 0$, and therefore $\mathfrak{g} = \mathfrak{g}'$

This concludes the proof that $s \in \mathfrak{g}$ and $n \in \mathfrak{g}$. $\square$

Proof of the theorem

Step 1 Since $\mathfrak{g}$ is semisimple, the adjoint representation is faithful, and we view $\mathfrak{g}$ as subalgebra of $\mathfrak{gl}(\mathfrak{g})$.

For X in $\mathfrak{g}$, let $X = s + n$ be the usual JD.

By the lemma, we have $s \in \mathfrak{g}$ and $n \in \mathfrak{g}$.

This shows existence and uniqueness of the absolute Jordan decomposition.

Step 2

Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional representation.

We get the commutative diagram

$$\begin{tikzcd}
\mathfrak{g} \arrow[r] \arrow[d, "\mathrm{ad}_{\mathfrak{g}} X"'] & \rho(\mathfrak{g}) \arrow[d, "\mathrm{ad}_{\rho(\mathfrak{g})} X"] \\
\mathfrak{g} \arrow[r] & \rho(\mathfrak{g})
\end{tikzcd}$$

We want to show $\rho(X)_s = \rho(s)$.

Since $\mathrm{ad}_{\rho(\mathfrak{g})} : \rho(\mathfrak{g}) \longrightarrow \mathfrak{gl}(V)$ is faithful, it suffices to show $\mathrm{ad}_{\rho(\mathfrak{g})}(\rho(X)_s) = \mathrm{ad}_{\rho(\mathfrak{g})}\rho(s)$.

$\forall X,$

$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[d, "\mathrm{ad}_{\mathfrak{g}} X"'] & \rho(\mathfrak{g}) \arrow[d, "\mathrm{ad}_{\rho(\mathfrak{g})} \rho(X)"] \\ \mathfrak{g} \arrow[r] & \rho(\mathfrak{g}) \end{tikzcd}$$

By functoriality of JD:

$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[d, "(\mathrm{ad}_{\mathfrak{g}} X)_s"'] & \rho(\mathfrak{g}) \arrow[d, "(\mathrm{ad}_{\rho(\mathfrak{g})} \rho(X))_s"] \\ \mathfrak{g} \arrow[r] & \rho(\mathfrak{g}) \end{tikzcd}$$

But $(\mathrm{ad}_{\mathfrak{g}} X)_s = \mathrm{ad}_{\mathfrak{g}} s$ by definition, and hence

$$(\mathrm{ad}_{\rho(\mathfrak{g})} \rho(X))_s = \mathrm{ad}_{\rho(\mathfrak{g})} \rho(s)$$

Hence we are done if we show

$$\mathrm{ad}_{\rho(\mathfrak{g})}\big(\rho(X)_s\big) = \big(\mathrm{ad}_{\rho(\mathfrak{g})}\,\rho(X)\big)_s$$

This follows the general theory of JD:

see the lecture on Cartan's criterion for a proof.

Step 3 Finally, we want to show that a morphism

$$\phi : \mathfrak{g} \to \mathfrak{g}'$$

to another semisimple Lie algebra, preserves absolute Jordan decompositions.

This follows by applying Step 2 to the representation

$$\mathrm{ad} \circ \phi : \mathfrak{g} \to \mathfrak{g}' \hookrightarrow \mathfrak{gl}(\mathfrak{g}')$$

$\square$