What is a motive?

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Abstract

The question in the title does not yet have a definite answer. One might even say that it is one of the most central, delicate, and difficult questions in arithmetic geometry. It is linked to important conjectures like the Hodge conjecture (Millenium problem), and Grothendieck's standard conjectures. In this talk I want to sketch what a motive *should* be, why we think they exist, and what problems arise in some of the approaches.

This talk borrows its title and much of its content from [2, 3]. First we introduce some terminology and notation.

Let k be field. We denote with $\mathcal{V}_k$ the category of (smooth, projective) varieties over k. Roughly speaking a variety is a geometric object, defined by finitely many polynomial equations in finitely many variables with coefficients in k. For example, the circle is defined by $x^2 + y^2 = 1$, and for $a, b \in k$ with $\Delta = -16(4a^3 + 27b^2) \neq 0$ the equation $y^2 = x^3 + ax + b$ defines a so-called *elliptic curve*.

Points on varieties correspond to solutions of the polynomial equations. Studying varieties is in general very hard. For example, the question whether the variety defined by $x^n + y^n = z^n$ over $\mathbb{Q}$ has non-trivial points is equivalent to Fermat's Last Theorem!

To obtain a better understanding of varieties, mathematicians devised cohomology theories, which assign vector spaces to varieties. For example, when k is a subfield of the field of complex numbers, the polynomial equations also define a complex manifold, and one can look at the singular cohomology of this manifold.

However, if k has characteristic $p > 0$, the above approach does not work, and we do not get a complex manifold, nor the corresponding singular cohomology groups. A. Weil made a very important conjecture that was an analogue of the Riemann hypothesis; but now for varieties over finite fields. It was clear that the conjectures would follow from the existence of a cohomology theory for varieties over finite fields with similar properties as the singular cohomology of complex manifolds. A. Grothendieck and P. Deligne constructed a candidate, namely étale cohomology. Using étale cohomology, P. Deligne was able to prove the Weil conjectures in the 70's of the previous century.

The thing with étale cohomology, however, is that it did not give one cohomology theory, but actually one for each prime $\ell \neq p$. The cohomology theory H_ℓ assigns to each variety X some vector spaces $H^i_\ell(X)$ over the field of ℓ-adic numbers: $\mathbb{Q}_\ell$.

In the meantime A. Grothendieck worked out another two cohomology theories, that also shared a lot of properties with singular cohomology and ℓ-adic étale cohomology:

- algebraic de Rham cohomology, which is comparable to the de Rham cohomology of complex manifolds;
- crystalline cohomology, which fills the gap for $\ell = p$ that the ℓ-adic étale cohomology theories left open.

The result was that in roughly two decades the stage changed drastically; where there was a shortage of suitable cohomology theories there is now an abundance of them:

- H_{sing}: singular cohomology, for varieties over a subfield of $\mathbb{C}$;
- H_ℓ: ℓ-adic étale cohomology, for varieties over a field of characteristic $p \neq \ell$;
- H_{dR}: algebraic de Rham cohomology, for all varieties;
- H_{cris}: crystalline cohomology, for varieties over perfect fields of characteristic $p > 0$.

Note some of the differences:

- The vector spaces $H^i_x(X)$ are vector spaces over different fields depending on $x \in \{\mathrm{sing}, \ell, \mathrm{dR}, \mathrm{cris}\}$;
- The cohomology theories do not work for arbitrary fields k; each puts its own restrictions on k.

Nevertheless, these cohomology theories also share a lot in common. For $x \in \{\mathrm{sing}, \ell, \mathrm{dR}, \mathrm{cris}\}$ and a variety X of dimension n, we have

- The rule $X \mapsto H^i_x(X)$ is a contravariant functor. That is, a map of varieties $X \to Y$ induces a linear map $H^i_x(Y) \to H^i_x(X)$;
- $\dim H^0_x(X) = 1 = \dim H^{2n}_x(X)$;
- For all $i < 0$ and $i > 2n$ we have $H^i_x(X) = 0$;
- For all i we have $\dim H^i_x(X) = \dim H^{2n-i}_x(X)$;

Moreover, suppose that X is a variety over a field k that allows for different choices of x (say k is a subfield of $\mathbb{C}$, so that we can take x to be sing, a prime ℓ, or dR). In this situation $\dim H^i_x(X)$ does not depend on the choice of x.

The above similarities give only a weak impression of the actual amount of features that these cohomology theories share. The similarities were formalised by FIXME, and cohomology theories that satisfy the formalised list of properties are nowadays called *Weil cohomology theories*.

1 Projective spaces over finite fields

Let $\mathbb{F}_q$ be a finite field with q elements. The projective space $\mathbb{P}^n(\mathbb{F}_q)$ is defined by $(\mathbb{F}_q^{n+1} - \{0\})/\mathbb{F}_q^*$. That is, we take an $(n+1)$-dimensional vector space, and look at the set of lines through the origin. (Indeed, a line through the origin consists of vectors that are scalar multiples of each other.)

From the definition we see that the number of points of $\mathbb{P}^n(\mathbb{F}_q)$ is equal to

$$(q^{n+1} - 1)/(q - 1) = 1 + q + q^2 + \ldots + q^n.$$

On the other hand, we may look at the ℓ-adic étale cohomology groups $H^i_\ell(\mathbb{P}^n(\mathbb{F}_q))$ for some prime ℓ not dividing q. First of all, these groups are 0 if i is odd, and 1-dimensional if i is even. Secondly, they carry a natural representation of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. The Frobenius element $\sigma \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is defined by $\overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q, x \mapsto x^q$.

Now the crucial fact is that σ acts as multiplication with q^i on $H^{2i}_\ell(\mathbb{P}^n(\mathbb{F}_q))$. A fancy way to say this, is that the eigenvalue of σ acting on $H^{2i}_\ell(\mathbb{P}^n(\mathbb{F}_q))$ is q^i. In particular the trace of σ acting on $H^\bullet_\ell(\mathbb{P}^n(\mathbb{F}_q)) = \bigoplus_i H^{2i}_\ell(\mathbb{P}^n(\mathbb{F}_q))$ is equal to the number of points of $\mathbb{P}^n(\mathbb{F}_q)$.

2 Jacobians

If X is a curve (a 1-dimensional variety), then we have another great hammer in our toolbox: the jacobian variety of X. The definition of this variety $J(X)$ goes beyond the scope of this talk, but it suffices to say that its points represent equivalence classes of line bundles (think: 1-dimensional vector bundles). The tensor product of line bundles equips this variety with the structure of a commutative group. Thus the jacobian is a mix of geometry and algebra. Finally, the construction is functorial, so that we get a functor

$$\begin{aligned} J\colon \{\text{curves}\} &\to \{\text{abelian varieties}\} \\ X &\mapsto J(X). \end{aligned}$$

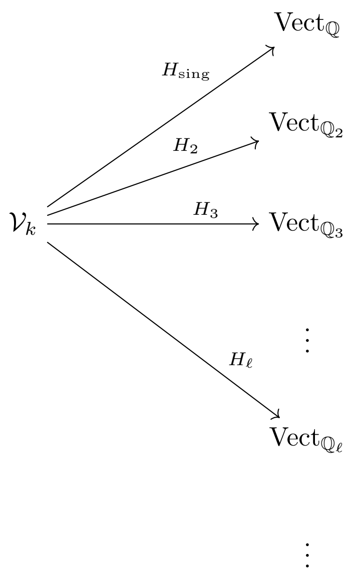
The category of abelian varieties is an additive category (and thus has an algebraic flavour). Moreover, the identity

$$H^1(X) \cong H^1(J(X))$$

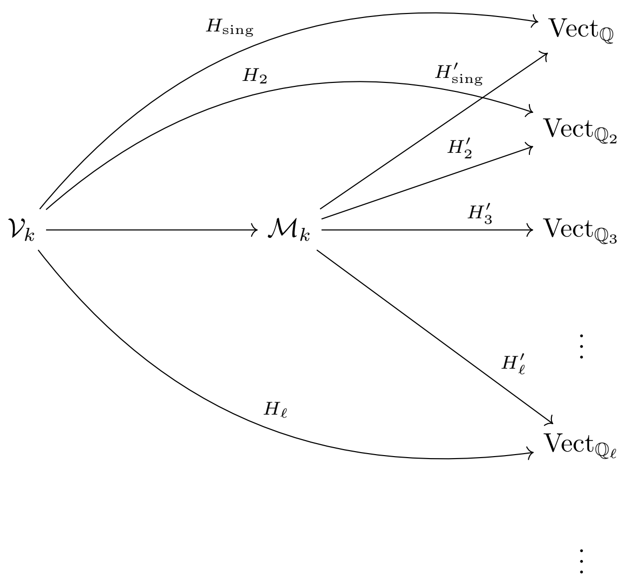
shows that the jacobian variety 'captures' the cohomological information in degree 1 of X.

3 So what should a motive be?

We have explained that there are several cohomology theories that share a lot of properties, but also have some incomparable differences.



The idea of motives is that there should be a category $\mathcal{M}_k$ that has an algebraic flavour sitting in between:



We stress that $\mathcal{M}_k$ has to have an algebraic flavour, because we do not want silly candidates as $\mathcal{M}_k = \mathcal{V}_k$. The algebraic flavour can be made more precise, by saying that $\mathcal{M}_k$ should behave as the category of representations of a group.

4 Conclusion

We have seen that, because varieties are pretty difficult to study, several cohomology theories have been invented to attach (linear) algebraic objects to varieties. These theories share a lot of properties, and therefore we hope that there is a universal theory.

> *Motives should be a universal Weil cohomology; on the one hand behaving as geometric objects, on the other hand having an algebraic flavour as category.*

Further, we saw that in the case of curves, the jacobian variety solves the problem. It is not known whether H^i is in general 'represented' by some object.

I also promised to tell a bit about current approaches to defining motives. Let me first of all say that there are a lot of approaches that I do not know anything about. However, most approaches fall in two classes.

- Either they focus on the nice categorical aspects that the category of motives should have. But then, it cannot be shown that the known cohomology theories factor throught this category of motives.
- The other approaches mainly focus on this 'universal property'. The make sure that all known cohomology theories factor through the category of motives. But then we cannot show that this category has the nice properties we want. For example, we then do not know if the H^i has a corresponding motivic object.

The Standard conjectures of Grothendieck imply that these approaches can be brought together. And, as Grothendieck wrote at the end of the paper [1] introducing these conjectures:

> Alongside the problem of resolution of singularities, the proof of the standard conjectures seems to me to be the most urgent task in algebraic geometry. (A. Grothendieck, 1969)

A Algebraic cycles & the Standard conjectures

An important rôle in the study of motives is played by algebraic cycles (roughly speaking: formal sums of closed subvarieties). Since cohomology is functorial, a map of varieties induces a map on cohomology. However, all these Weil cohomologies also come with a so called *cycle class map*. This map sends subvarieties of X to elements in the cohomology $H(X)$. Usually, the cohomology comes with some group action (e.g., by a Galois group), and the image of the cycle class map is always in the subspace that is invariant under the group action.

If one takes two varieties X and Y, and an algebraic cycle Z on $X \times Y$, then the cycle class map gives an element γ_Z of $H(X \times Y) = H(X) \otimes H(Y)$. One can then view γ_Z as a map from $H(X)$ to $H(Y)$ (using Poincaré duality).

This construction generalises the functoriality that I just mentioned. Given a map $f\colon X \to Y$, we can view the graph Γ_f of f as an algebraic cycle on $X \times Y$, namely $\Gamma_f = \{(x, f(x)) | x \in X\}$.

One approach to motives (falling in the second class mentioned in the Conclusion) is to enlarge the set of morphisms between varieties by declaring all algebraic cycles on $X \times Y$ to be morphisms from X to Y. Afterwards, one also formally adds objects, so that all idempotent morphisms have kernels.

The projection of $H(X) = \oplus_{i=0}^{2n} H^i(X)$ onto $\oplus_{i \neq k} H^i(X)$ is and idempotent morphism with kernel $H^k(X)$. The Standard conjectures say that this morphism is induced by an algebraic cycle on $X \times X$. (This is actually a weak version of one part of the Standard conjectures.)

References

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[3] James Milne. What is a motive?, March 2009. Version 1.02. Available at: http://www.jmilne.org/math/xnotes/mot.html.