Weil–Deligne representations

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December 11, 2014

1 Prolegomena

Let G be a group. Let K be a field. Let $\mathrm{Rep}_k(G)$ denote the category of representations of G over k. We can restrict to certain subcategories:

- the category of finite-dimensional representations;
- if k and G are endowed with a topology, the category of continuous representations;
- if G is endowed with a topology, the category of *smooth* representations.

Let us recall a generalisation of the definition of a smooth representation. Assume that G is endowed with a topology (e.g., G is locally profinite). A representation (ρ, V) of G is smooth if for every $v \in V$ the stabalizer $\mathrm{Stab}_v \subset G$ is open.

Let $i\colon K \to L$ be a field extension. There is an "extension of scalars" functor

$$\mathrm{Rep}_K(G) \longrightarrow \mathrm{Rep}_L(G)$$

that is compatible with the aforementioned restrictions to subcategories. (In the case of continuous representations, we obviously need i to be continuous.) If i is an isomorphism, the functor is an equivalence of categories.

The crucial remark is that the notion of smooth representation does not depend on a topology on the coefficient field. In particular, though the topologies on $\mathbb{C}$ and $\bar{\mathbb{Q}}_\ell$ are incompatible, the categories $\mathrm{Rep}_{\mathbb{C}}(G)$ and $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(G)$ are equivalent.

1.1 Structure of the Galois group of p-adic fields

Let p be a prime number. Let F be a p-adic field, with residue field of cardinality q. Let G_F denote the absolute Galois group of F. Let W_F be the Weil group of F. Let $I_F \subset W_F$ denote the inertia group. Let $P_F \subset I_F$ denote the wild inertia group. Recall that $v_F\colon W_F/I_F \to \mathbb{Z}$ is an isomorphism, where 1 corresponds with the class of geometric Frobenius elements (i.e., those reducing to $\phi = (x \mapsto x^{-q})$). We write $\|x\| = q^{-v_F(x)}$ Recall that I_F/P_F is canonically isomorphic to $\lim_{(n,p)=1} \mu_n$ (n ranges over the positive integers coprime to p). The action of $W_F/I_F \cong \mathbb{Z}$ on I_F/P_F is given by $wiw^{-1} = i^{\|w\|}$.

Let $\ell \neq p$ be a prime number. There is a canonical map $t_\ell\colon \lim_{(n,p)=1} \mu_n \to \lim_n \mu_{\ell^n}$ projecting I_F/P_F to its maximal ℓ-adic quotient. We write $\mathbb{Z}_\ell(1)$ for

$\lim_n \mu_{\ell^n}$. Note that $\mathbb{Z}_\ell(1)$ is a free $\mathbb{Z}_\ell$ module of rank 1, on which W_F/I_F acts via $wxw^{-1} = x^{\|w\|}$, for $x \in \mathbb{Z}_\ell(1)$ and $w \in W_F/I_F$.

Write $\mathbb{Z}_\ell(n)$ for the Galois module $\mathbb{Z}_\ell(1)^{\otimes n}$, if $n \geq 0$. If $n < 0$, then $\mathbb{Z}_\ell(n)$ denotes $\mathbb{Z}_\ell(-n)^\vee$. If K is a field extension of $\mathbb{Q}_\ell$, and V is a K-vector space (with an action of W_F/I_F), then $V(n)$ denotes $V \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(n)$.

1.2 A natural source of Galois representations

Let X/F be a projective smooth variety over F.

Remark 1.1. A short recap of what a projective smooth variety is.

The adjective projective indicates that X is a closed subspace of projective space, defined by homogeneous polynomial equations with coefficients in F.

The adjective smooth means that the Jacobian matrix of partial derivatives of these equations is a matrix of full rank.

The ℓ-adic étale cohomology

$$H^i_\ell = H^i_{\text{ét}}(X_{\bar{F}}, \mathbb{Q}_\ell) = \lim_{n \in \mathbb{Z}_{>0}} H^i_{\text{ét}}(X_{\bar{F}}, \mathbb{Z}/\ell^n\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

is a finite dimensional $\mathbb{Q}_\ell$-vector space that is naturally endowed with a continuous representation of G_F.

Remark 1.2. In general H^i_ℓ is not smooth. If it were smooth, then an open subgroup of G_F would act trivial. If $\dim H^i_\ell > 0$, this is only true for $i = 0$.

However, if we restrict to the Weil group W_F, we do not only change the group but also get a more permissive topology. In particular, the inertia group I_F is open in the Weil group.

Since H^i_ℓ is finite-dimensional, being smooth is equivalent with having an open kernel (i.e., an open subgroup acting trivially). In particular H^i_ℓ is smooth (as representation of W_F) if and only if an open subgroup of I_F acts trivially.

In other words, if after some finite (ramified) extension E/F, the inertia group I_E acts trivially. One says that H^i_ℓ as representation of G_E is *unramified*, and as representation of G_F is *potentially unramified*.

If X/F has good reduction, then H^i_ℓ is unramified, hence smooth. Conversely, if X/F is an abelian variety, and H^1 is unramified, then X/F has good reduction (this is the criterion of Néron–Ogg–Shafarevich).

Having good reduction, means that one can give a set of equations defining X, such the equations have integral coefficients, and such that after reducing these equations modulo the prime of F they define a smooth variety over the residue field.

2 Motivation for Weil–Deligne representations

Rougly speaking, a Weil–Deligne representation is

- a representation (ρ, V) of W_F;
- together with a nilpotent element $N \in \operatorname{End}(V)$;
- and some conditions that we make precise later.

To see why this definition makes sense we prove the ℓ-adic monodromy theorem of Grothendieck.

Theorem 2.1 (ℓ-adic monodromy theorem, Grothendieck)**.** *Let K be an extension of $\mathbb{Q}_\ell$. Let (ρ, V) be a finite-dimensional representation of W_F over K.*

Then there exists an open index subgroup $H \subset I_F$, such that $\rho(x)$ is unipotent for all $x \in H$.

Proof. Let d denote the dimension of V. Fix a basis of V, providing identifications $V \cong K^d$, and $\mathrm{GL}(V) \cong \mathrm{GL}_d(K)$. Let K^+ be the ring of integers of K, that is, the integral closure of $\mathbb{Z}_\ell$ in K.

Denote with K_i the multiplicative groups $1 + \ell^i \mathrm{GL}_d(K^+)$. These are open subsets of $\mathrm{GL}_d(K)$. By the continuity of ρ, we see that the image of an open subgroup H of W_F is contained in K_2. (We need this at the end of the proof.)

The image of H is a pro-ℓ-group (i.e., a projective limit of finite ℓ-groups). Recall that $I_F \twoheadrightarrow \prod_{\ell' \neq p} \mathbb{Z}_{\ell'}$ (with kernel, the wild ramification group P_F). Thus $\rho|_{I_F \cap H}$ factors via the projection $t_\ell \colon I_F \to \mathbb{Z}_\ell(1)$.

Write X for $\log(\rho(x)) = \sum_{j \in \mathbb{Z}_{\geq 0}} (-1)^{j-1} \frac{(\rho(x)-1)^j}{j}$. To see that the series converges, recall that $\rho(x)$ is in K_2, that is, $\rho(x)$ is $1 \mod \ell^2$; hence each term $(-1)^{j-1} \frac{(\rho(x)-1)^j}{j}$ is divisible by ℓ^j, which implies convergence.

Since $\rho|_{I_F \cap H}$ factors via $t_\ell \colon I_F \to \mathbb{Z}_\ell(1)$, we get the relation

$$\log(\rho(wxw^{-1})) = \log(\rho(x)^{\|w\|}) = \|w\| \log(\rho(x)) = \|w\| X,$$

for $x \in I_F$ and $w \in W_F/I_F$, which implies that for all $w \in W_F/I_F$ the matrices X and $\|w\|X$ are conjugate. Let $a_i(X)$ be the i-th symmetric polynomial expression in the roots of the characteristic polynomial of X. (So it is the $d-i$-th coefficient of the characteristic polynomial.) Then $a_i(X) = a_i(\|w\|X) = \|w\|^i a_i(X)$. Since the action of W_F/I_F on $\mathbb{Z}_\ell(1)$ is a free action (the extension of F generated by $\mathbb{Z}_\ell(1)$ is infinite), and $\mathrm{Aut}(\mathbb{Z}_\ell(1)) \cong \mathbb{Z}_\ell^*$ has a finite torsion subgroup, we can choose w such that it does not act torsion, which implies $a_i(X) = 0$, for $i > 0$. Consequently X is nilpotent. In particular $\rho(x) = \exp(X)$ is unipotent. □

Remark 2.2. A similar theorem is true when one replaces W_F by G_F. The proofs have a very similar flavour. If I am correct, theorem 2.1 implies the version for G_F, since unipotent subgroups are closed in the image of G_F, and $W_F \subset G_F$ is dense.

Corollary 2.3. *There exists a unique nilpotent operator $N \in \mathrm{End}_K(V)(-1)$, such that $\rho(x) = \exp(t_\ell(x)N)$, for all x in some open subgroup of I_F.*

Proof. Fix some $x_0 \in H \cap I_F$, such that $t_\ell(x_0)$ is non-trivial, and put $N = t_\ell(x_0)^{-1} \log(\rho(x_0))$. By theorem 2.1, N is a nilpotent element of $\mathrm{End}_K(V)(-1)$.

Recall that $\rho|_{H \cap I_F}$ factors via t_ℓ as some continuous representation σ. The continuous representation $\mathbb{Z}_\ell(1) \to \mathrm{GL}_K(V), x \mapsto \exp(xN)$, coincides with σ on $t_\ell(x_0)$, hence on $t_\ell(x_0)\mathbb{Z}_\ell(1)$. This proves existence of N. The uniqueness is immediate from the definition (take logarithms on both sides of $\rho(x) = \exp(t_\ell(x)N)$, and choose x such that $t_\ell(x)$ is non-trivial (hence invertible) in $\mathbb{Z}_\ell(1)$). □

3 Weil–Deligne representations

It is time for a proper definition of Weil–Deligne representations.

Definition 3.1. *A* Weil–Deligne representation *over K is a triple (ρ, V, N), where*

- *(ρ, V) is a finite-dimensional smooth representation of W_F over K;*
- *$N \in \mathrm{End}_K(V)$ is a nilpotent endomorphism,*

such that for all $x \in W_F$ the condition

$$\rho(x)N\rho(x)^{-1} = \|x\|N$$

holds. (Alternatively, N is a Galois-invariant element of $\mathrm{End}_K(V)(-1)$.)

A morphism of Weil–Deligne representations $(\rho_1, V_1, N_1) \to (\rho_2, V_2, N_2)$ is a map of representations $f\colon (\rho, V) \to (\rho_2, V_2)$, such that $f \circ N = N_2 \circ f$.

It is immediate from the definition that the category $\mathrm{WDRep}_K(W_F)$ does not depend on a topology on K. Just like in the prolegomena, we can therefore (non-canonically!) identify $\mathrm{WDRep}_{\mathbb{C}}(W_F)$ and $\mathrm{WDRep}_{\bar{\mathbb{Q}}_\ell}(W_F)$.

Remark 3.2. Corollary 2.3 provides us with a method to attach a Weil–Deligne representation to each finite-dimensional continuous representation of the Weil group.

Since (ρ, V) is not smooth in general, the naive approach $(\rho, V) \mapsto (\rho, V, N)$ does not work. We have to change the representation a bit.

What does work is taking a Frobenius element $\Phi \in W_F$, and defining a twist of ρ via

$$\rho_\Phi(\Phi^a x) = \rho(\Phi^a x)\exp(-t_\ell(x)N), \qquad a \in \mathbb{Z}, x \in I_F.$$

Lemma 3.3. *The triple (ρ_Φ, V, N) defines a Weil–Deligne representation.*

Proof.

- We check that for all $x \in W_F$ the condition

  $$\rho_\Phi(x)N\rho_\Phi(x)^{-1} = \|x\|N$$

  holds. Expanding the definition, this amounts to checking

  $$\rho(x)N\rho(x)^{-1} = \|x\|N \tag{3.4}$$

  because the inner exponentials commute with N, as the exponents are multiples of N.

  By the uniqueness of N, and its construction in theorem 2.1 and corollary 2.3 the formula (3.4) holds. (See Del-66, of Deligne's Antwerp paper.)

- This implies that $\rho_\Phi\colon W_F \to \mathrm{GL}(V)$ is a homomorphism, so that (ρ_Φ, V) is actually a representation of the Weil group.

  $$\begin{aligned}\rho_\Phi(\Phi^a x\Phi^b y) &= \rho_\Phi(\Phi^{a+b}(\Phi^{-b}x\Phi^b)y)\\ &= \rho(\Phi^{a+b}(\Phi^{-b}x\Phi^b)y)\exp(-t_\ell((\Phi^{-b}x\Phi^b)y)N)\\ &= \rho(\Phi^a x\Phi^b y)\exp(-(t_\ell(\Phi^{-b}x\Phi^b) + t_\ell(y))N)\\ &= \rho(\Phi^a x)\rho(\Phi^b y)\exp(-\|\Phi^{-b}\|t_\ell(x)N)\exp(-t_\ell(y)N)\\ &= \rho(\Phi^a x)\rho(\Phi^b y)\exp(-t_\ell(x)N)^{\|\Phi^{-b}\|}\exp(-t_\ell(y)N)\\ &= \rho(\Phi^a x)\exp(-t_\ell(x)N)\rho(\Phi^b y)\exp(-t_\ell(y)N).\end{aligned}$$

  In the last step we used the previous point.

- By corollary 2.3 it is trivial on some open subgroup of I_F. Hence (ρ_Φ, V) is smooth.

□

Theorem 3.5. *The functor*

$$\begin{aligned} (\text{—})_{\mathrm{WD}} \colon \operatorname{Rep}_K(W_F) &\longrightarrow \operatorname{WDRep}_K(W_F) \\ (\rho, V) &\longmapsto (\rho_\Phi, V, N) \end{aligned}$$

gives an equivalence of categories between the category of finite-dimensional continuous representations of the Weil group, and Weil–Deligne representations.

Proof. Lemma 3.3 shows that the functor is well-defined on objects. Let $f\colon (\rho_1, V_1) \to (\rho_2, V_2)$ be a map of W_F-representations. Then $f \circ N_1 = N_2 \circ f$, by the uniqueness of the monodromy operators (expand the N_i as logarithms, and the relation is obvious). By the same argument, one then finds $f \circ \rho_1(x) = \rho_2(x) \circ f$, for all $x \in W_F$. Consequently, $(\text{—})_{\mathrm{WD}}$ is a faithful functor.

To show that it is essentially surjective, observe that if (ρ, V, N) is a Weil–Deligne representation, then (ρ^Φ, V) with

$$\rho^\Phi(\Phi^a x) = \rho(\Phi^a x) \exp(t_\ell(x) N)$$

is a continuous representation of W_F. Indeed, ρ^Φ is a homomorphism by a similar argument as that in lemma 3.3, and it is continuous because ρ is continuous, as well as $I_F \to \mathrm{GL}_K(V), x \mapsto \exp(t_\ell(x)N)$. The uniqueness of the monodromy operator implies that N is the monodromy operator associated with (ρ^Φ, V). That the functor is full is now an analogous argument to that of faithfulness. □

Remark 3.6.

- We have not yet indicated whether the functor depends on our choice of Φ and t. It does, but only up to a natural automorphism of the identity functor. We leave this as an exercise to the reader.
- We should note that the operations of tensor product and dual in the category $\operatorname{WDRep}_K(W_F)$ are not defined as one might naively do.

  If one declares the above functor to be a tensor functor, one computes

$$\begin{aligned} (\rho_1, V_1, N_1) \otimes (\rho_2, V_2, N_2) &= (\rho_1 \otimes \rho_2, V_1 \otimes V_2, N_1 \otimes 1 + 1 \otimes N_2) \\ (\rho, V, N)^\vee &= (\rho^\vee, V^\vee, -N^\vee) \end{aligned}$$

  The reason for the formulas for the monodromy operators is

$$\begin{aligned} \log(\rho_1(x_0) \otimes \rho_2(x_0)) &= \log(\rho_1(x_0) \otimes 1 + 1 \otimes \log(\rho_2(x_0)) \\ \log(\rho_1(x_0)^{\vee,-1}) &= -\log(\rho_1(x_0))^\vee \end{aligned}$$

4 Semisimple and Φ-semisimple objects

By abstract nonsense, the functor of theorem 3.5 transfers semisimple objects to semisimple objects. However, there is a big catch to this. In the literature, and in my eyes this is very poor choice of terminology, a Weil–Deligne representation (ρ, V, N) is called *semisimple* if (ρ, V) is semisimple as representation of W_F. Using this notion, one calls a representation (ρ, V) of W_F Φ-*semisimple* if the attached Weil–Deligne representation $(\rho, V)_{\mathrm{WD}}$ is semisimple. In other words, if we postcompose $(—)_{\mathrm{WD}}$ with

$$\begin{aligned} \mathrm{WDRep}_K(W_F) &\longrightarrow \mathrm{Rep}_K(W_F) \\ (\rho, V, N) &\longmapsto (\rho, V) \end{aligned} \tag{4.1}$$

(which is not its inverse!), and we take the inverse image of the class of semisimple objects, then we obtain the Φ-semisimple representations of W_F.

It is true (and obvious from eq. (4.1)) that every categorically semisimple Weil–Deligne representation is semisimple.

Finally, one obtains the rather trivial corollary to theorem 3.5 that there is a canonical bijection between isomorphism classes of

- n-dimensional, Φ-semisimple, continuous representations of W_F;
- n-dimensional, semisimple, Weil–Deligne representations of W_F.

5 Conclusion

We end the talk with a couple of conjectures that are not directly related to the Langlands program, but very much involve Weil–Deligne representations.

Return to the situation where X/F is a smooth projective variety. Let ℓ and ℓ' be two primes different from p. Using the above theory we can attach a Weil–Deligne representation to the ℓ-adic cohomology H^i_ℓ. This is an object $(H^i_\ell)_{\mathrm{WD}}$ in $\mathrm{WDRep}_{\mathbb{Q}_\ell}(W_F)$.

Conjecture 5.1 (C_{WD}, Fontaine (1994))**.** *There is a Weil–Deligne representation H over $\mathbb{Q}$, such that $H \otimes \mathbb{Q}_\ell \cong (H^i_\ell)_{\mathrm{WD}}$ for all $\ell \neq p$.*

As a corollary to this conjecture, if we choose an embedding $i\colon \mathbb{Q}_\ell \to \mathbb{C}$, then the isomorphism class of $(H^i_\ell)_{\mathrm{WD}} \otimes_i \mathbb{C}$ does not depend on i or ℓ.

As final conclusion, a quote from Matt Emerton on MathOverflow:

> [F]rom the point of view of Galois representations, the point is that continuous Weil group representations on a complex vector space, by their nature, have finite image on inertia.
>
> On the other hand, while a continuous ℓ-adic Galois representation of $G_{\mathbb{Q}_p}$ (with $\ell \neq p$ of course) must have finite image on wild inertia, it can have infinite image on tame inertia. The formalism of Weil–Deligne representations extracts out this possibly infinite image, and encodes it as a nilpotent operator (something that is algebraic, and doesn't refer to the ℓ-adic topology, and hence has a chance to be independent of ℓ).