Twisted sheaves

Notes by Johan Commelin Friday, the 24th of November, 2017

1. Introduction

Recall that our final goal is the following statement:

> *Let S and S' be two complex algebraic K3 surfaces. Let $\phi\colon \mathrm{H}^2(S,\mathbb{Q}) \to \mathrm{H}^2(S',\mathbb{Q})$ be a Hodge isometry. Then ϕ is algebraic (i.e., induced by an algebraic correspondence on $S \times S'$).*

We will now give a rough sketch of Huybrechts's proof.

(a) Use lattice theory and moduli theory of K3 surfaces (specifically, the fact that the period map is surjective) to reduce to the case where ϕ is a *cyclic isogeny*.
(b) Show that S' is the (coarse) moduli space of *twisted* sheaves on S (depending on certain parameters).
(c) Show that ϕ corresponds with the Fourier–Mukai transform whose kernel is the (quasi)-universal sheaf $\mathcal{E}$ on $S \times S'$.
(d) Conclude that the algebraic correspondence that we are looking for is the twisted Chern class of $\mathcal{E}$ (up to a Todd class).

This sketch is as precise as it is short. In this talk we will start the work of making it precise. We will explain what is meant with the twisted sheaves mentioned in part (b). The following table explains what we know, and what we want to know.

	sheaves	*(semi)stable sheaves*	*moduli of sheaves*
untwisted	OK	(2)	(3)
twisted	(4)	(5)	(6) Goal

(2) Stability conditions
(3) Moduli of sheaves (Huybrechts–Lehn)
(4) Twisted sheaves (Căldăraru, Lieblich)
(5) Twisted stability conditions
(6) Moduli of twisted sheaves (Simpson, Yoshioka, Lieblich)

And then we need to apply this to our problem at hand:

(7) Application to K3 surfaces (Yoshioka's theorem)
(8) Twisted K3 surfaces, B-fields, twisted Hodge structures
(9) From Hodge isometries to twisted Fourier–Mukai equivalences
(10) Epic win!

2. Moduli of sheaves – abstract nonsense

Let $\star$ be a scheme, and let $X/\star$ be a scheme over $\star$. The functor $T/\star \mapsto \mathrm{Sh}(X_T)$ defines a (non-algebraic!) stack over $\star$. (The only thing that we are saying here is that one can glue sheaves together, etc.) It would make sense to call this stack $\mathrm{Sh}_{X/\star}$—the stack of sheaves on X. In the game of "moduli of sheaves" one seeks to identify substacks of $\mathrm{Sh}_{X/\star}$ that are "well-behaved": best of all a smooth proper scheme, but at least an Artin stack. To get something slightly more reasonable, we look at $\mathrm{Mod}_{X/\star}$, the stack defined by $T/\star \mapsto \mathcal{O}_{X_T}$-Mod.

Let us look at one particular example of a "well-behaved" substack, that we may pretend to understand. We consider the substack $P_{X/\star}$ defined by $T/\star \mapsto \{\text{line bundles on } X_T\}$. Recall that a line bundle on a scheme Y is the same as a $\mathbb{G}_\mathrm{m}$-torsor, which we know to be the same as a map $Y \to \mathrm{B}\mathbb{G}_\mathrm{m}$. This shows that we have an action of $\mathrm{B}\mathbb{G}_\mathrm{m}$ on $P_{X/\star}$, as follows: $(L_T, F) \mapsto \pi^*(L_T) \otimes F$. Here $\pi \colon X_T \to T$ is the canonical projection, while L_T and F are line bundles on T and X_T respectively. The quotient $P_{X/\star}/\mathrm{B}\mathbb{G}_\mathrm{m}$ is well-understood: it is $\mathrm{Pic}_{X/\star}$. Under suitable conditions this turns out to be a representable sheaf, the Picard scheme, whereas $P_{X/\star}$ is merely an Artin stack. Depending on your point of view, this is an improvement.

In one respect, it is a failure. One might hope that elements of $\mathrm{Pic}_{X/\star}(T)$ may be *represented* by line bundles on X_T. In other words, on might hope that there is a section $\mathrm{Pic}_{X/\star} \to P_{X/\star}$. It turns out that in general, this is not the case. Let us investigate what is going on.

Consider the following diagram.

$$\begin{array}{ccc} P_{X/\star} & \longrightarrow & \star \\ \big\downarrow & & \big\downarrow \\ \mathrm{Pic}_{X/\star} & \longrightarrow & ? \end{array}$$

By working with 2-stacks, and taking $? = \mathrm{B}^2\mathbb{G}_\mathrm{m} = \star/\mathrm{B}\mathbb{G}_\mathrm{m}$, we can make this square into a pullback diagram. To understand the map $P_{X/\star} \to \mathrm{Pic}_{X/\star}$, we want to understand the map $\star \to \mathrm{B}^2\mathbb{G}_\mathrm{m}$. What does this leave us with? In other words, what is the modular interpretation of $\mathrm{B}^2\mathbb{G}_\mathrm{m}$; what does it represent? The answer is: $\mathbb{G}_\mathrm{m}$-*gerbes*.

3. The inertia stack and gerbes

Let $\star$ be a scheme (or ringed topos), and let $S/\star$ be a stack. In earlier talks we have considered the stack $S \times_{S \times S} \times S$ over S, the pullback of the diagonal $S \to S \times S$ along itself. We denote this stack with $I(S)$, and call it the *inertia stack*. This stack is a sheaf over S (in other words, it does not add stackiness); indeed, $I(S)$ represents the presheaf $T \mapsto \mathrm{Aut}(T)$, showing that $I(S)$ is a group stack.

If F is a sheaf on S, we get a right action of $I(S)$ on F: given $T \to S$, and $\phi \in \mathrm{Aut}(T)$ we get a map $\phi^* \colon F(T) \to F(T)$.

Observe that there is a natural source of group stacks on S, namely those coming from $\star$: Write $\pi \colon S \to \star$ for the structure map, and let $G/\star$ be a group scheme. Then we may consider $\pi^* G$. This will be a key ingredient in the definition of a G-gerbe. But first we need to define what a gerbe is in general.

Gerbes should be thought of as a higher-categorical analogue of torsors. (This is reflected by the fact that for any commutative group scheme G the cohomology group $\mathrm{H}^1(T, G)$ classifies G-torsors, whereas $\mathrm{H}^2(T, G)$ classifies G-gerbes. Of course, this statement does not make sense right now, because we have not yet defined G-gerbes.)

3.1 Definition. A stack $S/\star$ is a *gerbe* if for every $U/\star$ there exists a covering $U' \to U$ such that $S_{U'} \neq \varnothing$, and for every $U/\star$ and objects $s_1, s_2 \in S_U$ there exists a covering $U' \to U$ such that $s_{1,|U'} \cong s_{2,|U'}$.

These conditions in the definition of a gerbe should be thought of as saying that a stack is locally trivializable.

3.2 Definition. Let G be a group scheme over $\star$. A *G-gerbe* is a stack $S/\star$ together with an isomorphism $I(S) \to \pi^*G$.

We claim that $\star \to \mathrm{B}^2\mathbb{G}_\mathrm{m}$ is a $\mathbb{G}_\mathrm{m}$-gerbe. Indeed, let us compute the inertia stack: We have $\star \times_{\mathrm{B}^2\mathbb{G}_\mathrm{m}} \star \cong \mathrm{B}\mathbb{G}_\mathrm{m}$ and $\star \times_{\mathrm{B}\mathbb{G}_\mathrm{m}} \star \cong \mathbb{G}_\mathrm{m}$. Therefore, for every $T \to \mathrm{B}^2\mathbb{G}_\mathrm{m}$ the pullback $\star \times_{\mathrm{B}^2\mathbb{G}_\mathrm{m}} T$ is a $\mathbb{G}_\mathrm{m}$-gerbe over T. By abstract nonsense, the converse is true, and $\mathrm{B}^2\mathbb{G}_\mathrm{m}$ is the moduli stack of $\mathbb{G}_\mathrm{m}$-gerbes.

Finally, note that $\mathrm{Hom}(T, \mathrm{B}^2\mathbb{G}_\mathrm{m}) = \mathrm{H}^2(T, \mathbb{G}_\mathrm{m})$, almost by definition. The cohomology group $\mathrm{H}^2(T, \mathbb{G}_\mathrm{m})$ is called the *cohomological Brauer group*.

4. Twisted sheaves

For a pretty general treatment of twisted sheaves, see [L].

4.1 Definition. A twisted sheaf on X is a map $X \to \mathrm{Mod}_\star/\mathrm{B}\mathbb{G}_\mathrm{m}$.

Let us unwind this definition. We have a diagram similar to the one above.

$$\begin{array}{ccccc} \mathcal{G} & \longrightarrow & \mathrm{Mod}_\star & \longrightarrow & \star \\ \downarrow & & \downarrow & & \downarrow \\ T & \longrightarrow & \mathrm{Mod}_\star/\mathrm{B}\mathbb{G}_\mathrm{m} & \longrightarrow & \mathrm{B}^2\mathbb{G}_\mathrm{m} \end{array}$$

We see that $T \to \mathrm{Mod}_\star/\mathrm{B}\mathbb{G}_\mathrm{m}$ induces an $\mathcal{O}_\mathcal{G}$-module F on the $\mathbb{G}_\mathrm{m}$-gerbe $\mathcal{G}$. The important point is that F is not just any module: the map $\mathcal{G} \to \mathrm{Mod}_\star$ is $\mathrm{B}\mathbb{G}_\mathrm{m}$-equivariant. Unwinding what this means is that two $\mathbb{G}_\mathrm{m}$-actions on F coincide: the action coming from the inertia stack of $\mathcal{G}$, and the action coming from the $\mathcal{O}_\mathcal{G}$-module structure.

Finally, unwinding even further, we find recover a more hands-on definition:

Let X be a smooth projective scheme over an algebraically closed field k. (More generally, let X be a ringed topos.) Fix $\alpha \in \mathrm{H}^2(X, \mathbb{G}_\mathrm{m})$. There is a hypercovering $U_\bullet \to X$, and a cocycle $a \in \Gamma(U_2, \mathbb{G}_\mathrm{m})$ representing α.

4.2 Definition (Căldăraru). A twisted sheaf on X is a pair (F, g), where F is an $\mathcal{O}_{U_0}$-module and $g\colon (\pi_1^{U_1})^*F \to (\pi_0^{U_1})^*F$ is a gluing datum such that the coboundary $\delta g \in \mathrm{Aut}((\pi_0^{U_2})^*F)$ equals a.

We wrap up with the following general result.

4.3 Proposition (2.3.1.1 of [L]). *Let $X \to \star$ be a proper morphism of algebraic spaces that is locally of finite presentation. Let n be an integer that is invertible on X, and fix a μ_n-gerbe $\mathcal{X} \to X$. For each* $\mathrm{Spec}(R) \to \star$, *let $\mathcal{T}_{\mathcal{X}/\star}(R)$ be the groupoid of coherent $\mathcal{X}$-twisted sheaves on X_R that are flat over R. Then $\mathcal{T}_{\mathcal{X}/\star}$ is an algebraic stack over $\star$ that is locally of finite presentation.*

5. References

[L] Lieblich, Max. *Moduli of twisted sheaves.* Duke Mathematical Journal. Vol. 138, No. 1, 2007.