Periods (and why the fundamental theorem of calculus conjecturely is a fundamental theorem)

by J.M. Commelin Saturday, the 28th of November, 2015

0.1 abstract. — In 2001 M. Kontsevich and D. Zagier posed a conjecture on algebraic integrals, which rougly says that the theorem of Stokes (the fundamental theorem of calculus in higher dimensions) is the only non-trivial relation between such integrals. In this talk I will formulate this conjecture, and indicate how it relates to conjectures and research in other fields.

0.2 Protagonists. — This talk owes a lot to work of M. Kontsevich, D. Zagier, Y. André, A. Grothendieck, A. Connes, F. Brown, P. Cartier, (J. Ayoub, M. Nori, and more). For this talk I made use of [7, 6, 4, 2, 1, 5]

1 What is a period?

Let $f_1, \ldots, f_n$ be polynomials in $\mathbb{Q}[X_1, \ldots, X_d]$. We define:

$$Z(f_1, \ldots, f_n) = \left\{ x \subset \mathbb{R}^d \,\middle|\, f_i(x) \geq 0 \text{ for all } 1 \leq i \leq n \right\}$$

1.1 Definition. — A *rational algebraic set* is a subset of $\mathbb{R}^d$ of the form $Z(f_1, \ldots, f_n)$ where the f_i are polynomials in $\mathbb{Q}[X_1, \ldots, X_d]$. «

1.2 Definition. — A *rational function* in d variables is an element of $\mathbb{Q}(X_1, \ldots, X_d)$. «

1.3 Definition. — A *real period* is a real number $r \in \mathbb{R}$, such that there exists a positive integer $d \in \mathbb{Z}_{\geq 0}$, some rational algebraic set $D \subset \mathbb{R}^d$, and some algebraic function f in d variables, such that

$$r = \int_D f \, \mathrm{d}x_1 \cdots \mathrm{d}x_d .$$

A *period* is a complex number z, such that $\Re(z)$ and $\Im(z)$ are real periods. «

1.4 Example. — The following numbers are periods:

$$\begin{aligned}
\sqrt{2} &= \int_{2x^2 \leq 1} \mathrm{d}x \\
\log(2) &= \int_1^2 \frac{\mathrm{d}x}{x} \\
\pi &= \iint_{x^2+y^2 \leq 1} \mathrm{d}x \, \mathrm{d}y \\
\zeta(3) &= \int_{0 \leq x_1 \leq x_2 \leq x_3 \leq 1} \frac{\mathrm{d}x_1}{1-x_1} \frac{\mathrm{d}x_2}{x_2} \frac{\mathrm{d}x_3}{x_3} \\
& 2\int_{-b}^{b} \sqrt{1 + \frac{a^2x^2}{b^4 - b^2x^2}} \, \mathrm{d}x \qquad \text{perimeter of ellipse with radii } a \text{ and } b
\end{aligned}$$

More generally, all algebraic numbers are periods; logarithms of algebraic numbers are periods; $\zeta(n)$, for $n \in \mathbb{Z}_{\geq 2}$, or even “multiple zeta values” are periods.

On the other hand, it is conjectured that e and $1/\pi$ are not periods. «

Periods form a ring $\mathscr{P}$, since integrals can be added and multiplied (Fubini). The ring is countable, since we can enumerate polynomials and rational functions.

2 The period conjecture

2.1 Definition. — The *space of abstract periods*, $\mathcal{P}$, is the $\mathbb{Q}$-vector space generated by symbols (D, ω), where

» D is an algebraic subset of $\mathbb{R}^d$ for some $d \in \mathbb{Z}_{\geq 0}$; and
» ω is a rational d-form on D (*i.e.* $f\, \mathrm{d}x_1 \cdots \mathrm{d}x_d$ for some rational function f in d variables);

modulo the relations:

» (linearity) $\lambda(D, \omega) + \lambda'(D, \omega') \sim (D, \lambda\omega + \lambda\omega')$; and
» (variable substitution) $(\phi(D), \omega) \sim (D, \phi^*\omega \cdot |\det J_\phi|)$, for invertible rational functions $\phi \colon \mathbb{R}^d \to \mathbb{R}^d$ (where J_ϕ denotes the Jacobian matrix of partial derivatives of ϕ); and
» (fundamental theorem of calculus a.k.a. Stokes' theorem) $(\partial D, \omega|_{\partial D}) \sim (D, \mathrm{d}\omega)$, where ∂D is the boundary of D, and $\mathrm{d}\omega$ is the *exterior derivative* of ω. «

Let me emphasize that Stokes theorem in one variable really just says $\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$.

2.2 Conjecture (Kontsevich–Zagier). — *The natural evaluation map*

$$\begin{aligned} \mathcal{P} &\longrightarrow \mathscr{P} \\ (D, \omega) &\mapsto \int_D \omega \end{aligned}$$

is an isomorphism. «

Observe that this map is surjective by definition of $\mathscr{P}$.

This conjecture is almost innocent to state, but is of the same caliber as some of the Millenium problems. If it is true, it answers many questions about transcendence and algebraic dependence of numbers occuring naturally in arithmetic.

2.3 Reformulation in terms of algebraic geometry. — The definition of period in terms of algebraic sets and rational functions can be viewed as an algebro-geometric definition in a natural way. Let X be an algebraic variety over $\mathbb{Q}$. The complex solutions of the polynomials defining X form a complex manifold, denoted $X(\mathbb{C})$. By a theorem of de Rham, there is a perfect pairing between singular homology and de Rham cohomology:

$$\begin{aligned} \mathrm{H}_i^{\mathrm{sing}}(X(\mathbb{C}), \mathbb{C}) \times \mathrm{H}^i_{\mathrm{dR}}(X(\mathbb{C})) &\longrightarrow \mathbb{C} \\ (\gamma, \omega) &\longmapsto \int_\gamma \omega \end{aligned}$$

The singular homology group parameterises (classes of) subsets over which one can integrate; while the de Rham cohomology group parameterises (classes of) differential forms that can be integrated.

Since X is an algebraic variety over $\mathbb{Q}$, there exists a version of de Rham cohomology that yields a $\mathbb{Q}$-vector space. The above pairing then becomes

$$\begin{aligned} \mathrm{H}_i^{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Q}) \times \mathrm{H}^i_{\mathrm{dR}}(X) &\longrightarrow \mathbb{C} \\ (\gamma, \omega) &\longmapsto \int_\gamma \omega \end{aligned}$$

and complex numbers that are in the image of this map are called *periods* (of weight i) of X. Every period in the previous sense is the period of some X, and vice versa. (Technically, we need to consider *boundary divisors*, but I will ignore this for this talk.) Let us denote this set with $\mathscr{P}^i(X)$.

2.4 Definition. — The *field of periods* (of weight i) of X (or (X, D)) is the subfield $\mathbb{Q}(\mathscr{P}^i(X))$ of $\mathbb{C}$ generated by the periods of X (or (X, D)). «

The cohomology groups of X all carry natural representations by pretty big groups. Conjecturely, the images of all these representations should yield the same group. This group is called the *motivic Galois group* of X (for weight i).

Under the Hodge conjecture, it is isomorphic to the Mumford–Tate group $\mathrm{G}^i_{\mathrm{B}}(X)$. Under the Tate conjecture, it is isomorphic to the image of the Galois representation on étale cohomology: $\mathrm{G}^i_{\ell}(X)$. We will accept these hard conjectures, for the purposes of this talk; but let me say that there are good theories of motives that allow for unconditional definitions of the motivic Galois group.

The period conjecture by M. Kontsevich and D. Zagier, as stated above, is equivalent to the following (older) conjecture by A. Grothendieck (a generalisation of note 10 of [5]).

2.5 Conjecture (Grothendieck). — *For all algebraic varieties X and integers i*

$$\dim_{\mathbb{Q}} \mathrm{G}^i_{\mathrm{B}}(X) = \mathrm{tr.deg.}_{\mathbb{Q}} \mathbb{Q}(\mathscr{P}^i(X)).$$ «

3 Motives

We already briefly mentioned motives in the previous section. The entire idea behind motives goes back to A. Grothendieck, and is about searching for a universal cohomology theory for algebraic varieties. To each algebraic variety X one can associate a wealth of cohomology groups, that all behave very similar. A map $X \to Y$ of varieties, induces maps $\mathrm{H}(Y) \to \mathrm{H}(X)$ on these cohomology groups. A motive is more or less a summand of the cohomology of an algebraic variety that occurs as the kernel or image of such an induced map.

This brief and sketchy introduction can be made very precise, and recently[1] there has been a lot of progress in developing a good theory of motives. Let M be a motive occuring as summand of the cohomology of sum variety X. Just like we associated periods with X, we may now associate a natural subset $\mathscr{P}(M)$ of $\mathscr{P}(X)$ with M. In other words, it makes sense to speak about the periods of a motive.

Even more is true. The period conjecture claims that one can recover a motive M from the set $\mathscr{P}(M)$. (To be precise, one has to look count the periods with multiplicities, because there might be an irreducible motive occuring more then once in the decomposition of M into irreducible motives.) This means that it should be possible to define motives in terms of the algebra of periods. This is precisely what M. Kontsevich does in definition 23 of [6]. For technical reasons, one has to add the inverse of π to get a good theory of motives. Let $\hat{\mathscr{P}}$ be $\mathscr{P}[\frac{1}{2\pi i}] \subset \mathbb{C}$.

3.1 Definition (paraphrase of definition 23 of [6]). — A *framed motive* of rank $r \geq 0$ is an invertible $(r \times r)$-matrix $(P_{ij})_{1 \leq i,j \leq r}$ with coefficients in the algebra $\hat{\mathscr{P}}$, satisfying the equation a technical condition[2] on the coefficients.

The space of morphisms from one framed motive to another, corresponding to matrices

$$P_1 \in \mathrm{GL}_{r_1}(\hat{\mathscr{P}}) \quad \text{and} \quad P_2 \in \mathrm{GL}_{r_2}(\hat{\mathscr{P}}),$$

is defined as

$$\left\{ T \in \mathrm{Mat}_{r_2 \times r_1}(\mathbb{Q}) \,\middle|\, TP_1 = P_2 T \right\}.$$ «

4 Relation with theoretical physics

Let me first point out that I do not know anything at all about modern theoretical physics. My knowledge does not extend beyond elementary high school physics. Please accept my sincere apologies for my ignorance.

Nevertheless, let me to quote M. Kontsevich [6, §5.2] to point out that there seem to be very deep connections between the theory of periods and theoretical physics:

[1] Without going into the entire history, let me mention the work of V. Voevodsky, which is now pushed to the limit by J. Ayoub.

[2] This condition basically says that the matrix must occur as submatrix of the period matrix of an algebraic variety.

D. Broadhurst and D. Kreimer (see [3]) observed that all Feynman diagrams up to 7 loops in any QFT in even dimensions gives same numbers as appear in Drinfeld associator. It is not clear a priori why this happens. In any case one can see immediately from formulas that all constants are in fact periods.

4.1 CONJECTURE. — The motivic Galois group $\mathrm{G_B}$ acts (in homotopy sense) on the homotopy Lie algebra $\mathfrak{g}^*$ associated with the free massless theory in any dimension. In the case of even dimension the action factors through the quotient group GT as in [6, §3.4]. The action should be somehow related with the action on values of Feynman integrals.

In [4], P. Cartier seems to suggest that a better understanding of periods and the associated pro-algebraic groups would yield insight in fundamental physical constants such as the fine structure constant, and a "cosmic Galois group" acting on them.

Finally, for a long time people had conjectured that the periods (*resp.* motives) associated with QFTs would be multiple zeta values (*resp.* mixed Tate motives). However, a few years ago F. Brown proved that this is not the case. In my primitive understanding, I think this means that the second sentence in the quoted conjecture by M. Kontsevich is wrong. Apparently, a bigger class of motives is needed to describe QFTs.

5 Appendix on Feynmann graphs and amplitudes

Let G be a graph. We introduce some notation:

$V(G)$	the set of vertices of G
$E(G)$	the set of edge of G
$\deg(v)$	the degree of $v \in V(G)$, *i.e.,* the number of edges leaving v
$\pi_0(G)$	the set of connected components of G
$b_1(G)$	the first Betti number of G, equal to $\#E(G) - \#V(G) + \#\pi_0(G)$

Recall that a spanning tree $T \subset G$ is a subgraph of G such that $V(T) = V(G)$, $\#\pi_0(T) = 1$, and $b_1(T) = 0$.

5.1 DEFINITION. — The *graph polynomial* of G is the polynomial $\Psi_G \in \mathbb{Z}[\alpha_e | e \in E(G)]$ given by

$$\Psi_G = \sum_{T \subset G} \prod_{e \notin E(T)} \alpha_e,$$

where the sum is over all spanning trees $T \subset G$. «

Observe that Ψ_G is homogeneous of degree $b_1(G)$.

5.2 DEFINITION. — A graph G is called

» *physical* if $\deg(v) \leq 4$ for all $v \in V(G)$;
» *overall log-divergent* if $\#E(G) = 2b_1(G)$; and
» *primitive* if $\#E(\gamma) > 2b_1(\gamma)$ for every subgraph $\gamma \subsetneq G$. «

Let G be a physical overall log-divergent primitive graph. Choose a numbering of the edges $E(G) \cong \{1, \ldots, \#E(G)\}$. Let $\sigma(G)$ be the standard simplex on the edges of G, in other words,

$$\sigma(G) = \Big\{(t_1, \ldots, t_{\#E(G)}) \Big| \sum_e t_e = 1\Big\}.$$

Let Ω_G be the $(\#E(G) - 1)$-form given by

$$\Omega_G = \sum_{e=1}^{\#E(G)} (-1)^e \alpha_e \, \mathrm{d}\alpha_1 \wedge \ldots \wedge \widehat{\mathrm{d}\alpha_e} \wedge \mathrm{d}\alpha_{\#E(G)}.$$

The amplitude of G is the integral

$$I_G = \int_{\sigma(G)} \frac{\Omega_G}{\Psi_G^2}.$$

The integral converges by virtue of our assumptions (log-divergent, primitive) on G. It is immediate that I_G is a period. It is also this period of which people for a long time thought that it would always be a mixed zeta value; but F. Brown gaven example of a (planar) graph G such that I_G is not a mixed zeta value.

Once more: I see the relation between such graphs and Feynmann diagrams, but I do not know anything about how Feynmann diagrams play a role in theoretical physics. I would be very glad to learn this from the experts in the audience.

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