LIQUID VECTOR SPACES FOR COMPLEX GEOMETERS

JOHAN COMMELIN

1. Disclaimer

In this talk I will give an exposition of condensed mathematics and liquid vector spaces developed by Dustin Clausen and Peter Scholze. I will not present work of my own; but all mistakes are mine.

2. Goals of this talk

I have two aims with this talk.

(1) Show that liquid vector spaces aren't scary: a large class of them admits a down-to-earth description generalizing familiar objects from functional analysis.
    Slogan: liquid vector spaces are a good alternative to complete (locally convex) topological vector spaces.
(2) Give a teaser for why it is useful to work with liquid vector spaces: they unlock tools from sheaf theory and homological algebra that weren't available before in complex geometry.

This talk will not give many details or precise definitions. But I hope that it will be an appetizer that provides the motivation sit down for an elaborate main course.

3. Sources and other material

The following sources contain details and information that I am happily omitting.

- The three sets of lecture notes by Clausen and Scholze: "Condensed Mathematics" [2], "Analytic Geometry" [3], and "Complex Geometry" [1].
- The master thesis of Dagur Ásgeirsson.
- A detailed computation of a fundamental counterexample in the theory of liquid vector spaces is available at: `https://math.commelin.net/files/liquid_example.pdf`
- Several recorded lectures on Youtube: TODO (by Clausen and Scholze)

4. Condensed sets

Before turning to liquid vector spaces, we need to talk a bit about condensed sets. In this talk, I take an axiomatic approach.

Fact 4.1. Condensed sets exist. Every compact Hausdorff space is a condensed set.

Slightly more abstract and precise.

Fact 4.2. The category CHaus of compact Hausdorff spaces is a full subcategory of the category Cond of condensed sets. The category Cond has all limits and colimits.

These facts allow us to single out the "Hausdorff" condensed sets, which admit a rather elementary (and in my opinion, intuitive) description.

Definition 4.3. A monomorphism $X \to Y$ of condensed sets is *closed* if for every $K \in \mathrm{CHaus}$ mapping to Y the pullback $X \times_Y K$ is compact Hausdorff.

Definition 4.4. A condensed set X is *separated* if the diagonal $X \to X \times X$ is closed.

4.5. A related notion in category theory is that of *quasiseparated* objects. For condensed sets, these notions turn out to be the same. The terminology *quasiseparated* prevails in the literature.

Quasiseparated condensed sets are the same as *compactological spaces*, a notion introduced by Waelbroeck in the '70s.

Definition 4.6 ([4, Ch. III])**.** A *compactological space* is a set X equiped with a *compactology*, which consists of a topology and a bornology that are compatible in the way prescribed below. Recall that a bornology endows X with a collection of "small" subsets that satisfy the following conditions:

- every finite subset of X is small;
- finite unions of small subsets are small;
- subsets of small sets are small.

The topology and bornology form a compactology if they satisfy the following axioms:

- the closure of a small set is small;
- the closed small subsets are compact Hausdorff;
- the topology on X is the colimit topology of the closed small subsets.

A morphism of compactological spaces $X \to Y$ is a function that is continuous and sends small subsets of X to small subsets of Y.

Fact 4.7 ([3, Prop. 1.2])**.** The category of quasiseparated condensed sets is equivalent to the category of compactological spaces.

4.8. We can now give some examples of qs condensed sets:

- Compact Hausdorff spaces: all subsets are small
- Discrete sets: a subset is small iff finite
- $\mathbb{R}^n$: the small subsets are the bounded ones (use Heine–Borel)
- Topological $\mathbb{R}$-vector spaces: the small subsets are those that are contained in compact Hausdorff subsets.

4.9. Small warning: the underlying topology of a compactological space X does not have to be Hausdorff. The reason is that $X \times X$ does not carry the product topology, but its k-ification (aka the coreflection into CGWH).

4.10. What about the condensed sets that are not quasiseparated? Their existence is very important for the whole theory: they are the reason that Cond has nice categorical properties, which in turn is the reason that we can unlock the tools from sheaf theory and homological algebra.

All such condensed sets are quotients of compactological spaces. If X is a compactological space and $E \subset X \times X$ is compactological subspace that is an equivalence relation, then we can form X/E as condensed set. If E is a closed equivalence relation, then X/E is again compactological. If it is not closed, then we get one of the "mystery" objects.

5. Liquid vector spaces

To start things of, I will first describe quasiseparated liquid vector spaces. The whole theory depends on a parameter $0 < p \leq 1$. Feel free to pick $p = 1$ for some extra comfort.

Fact 5.1 ([1, Thm 2.14])**.** A qs condensed $\mathbb{R}$-vector space is *p-liquid* if for all $q < p$ and every compact $K \subset V$ there exists a compact *q-convex* subset of V containing K.

Definition 5.2. A compact $K \subset V$ is *q-convex* if for all $x_1, \ldots, x_n \in K$ and all $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ with $\sum |\lambda_i|^q \leq 1$ we have $\sum \lambda_i x_i \in K$.

The general definition of a (non-qs) p-liquid vector space looks a bit different. I will not give that definition in this talk, but I strongly recommend taking a look a the first four lectures of "Complex Geometry" [1], which contain a detailed account. Once again, the non-qs objects are quotients of the qs liquid vector spaces.

5.3. All complete locally convex topological vector spaces are p-liquid. In particular, all Banach spaces and Frechet spaces[1] are p-liquid. But there are many more p-liquid vector spaces. The category has very nice properties (which relies crucially on the fact that non-qs objects exist). Continuing our axiomatic approach, we list some of these properties below.

Fact 5.4 ([3, §VI])**.**

- The category of p-liquid vector spaces is an abelian category.
- It is a full subcategory of $\mathrm{Cond}(\mathbb{R})$, stable under all limits, colimits, and extensions.
- It has an internal Hom, and a tensor product, that are adjoint in the expected manner.
- The tensor product agrees with the tensor product of nuclear Frechet spaces. (Nuclear spaces are the objects in functional analysis where all "37" different topological tensor products agree.)
- There is a *liquidification* functor $\mathrm{Cond}(\mathbb{R}) \to \mathrm{Liq}_p$ which is left adjoint to the inclusion $\mathrm{Liq}_p \subset \mathrm{Cond}(\mathbb{R})$.

6. Quasicoherent liquid sheaves

6.1. Let U be some open subset of $\mathbb{C}^n$ (for the analytic topology). Then we can consider the ring of holomorphic function $\mathcal{O}(U)$ as condensed ring. It is liquid because it is a Frechet space. Thus it makes sense to speak of liquid $\mathcal{O}(U)$-modules, and hence of liquid $\mathcal{O}$-module sheaves.

Fact 6.2 ([1, Exc. 1 of §VI])**.** Consider an open subset $U \subset \mathbb{C}^n$ (for the analytic topology). Let $\mathcal{O}_U$ denote the structure sheaf (of holomorphic functions). A *quasicoherent* liquid sheaf, is a liquid $\mathcal{O}$-module sheaf $\mathcal{M}$ such that for every open polydisk $D \subset U$ the natural map

$$\mathcal{M}(D) \otimes_{\mathcal{O}(D)} \mathcal{O}|_D \to \mathcal{M}|_D$$

is an isomorphism.

6.3. In fact, one should really do all of this in the derived setting. And here it pays off to use the machinery of ∞-categories. One big benefit of working with ∞-categories, is that they can be glued. Indeed, the construction $U \mapsto C(U)$ is a sheaf of stable ∞-categories. (Being

[1]Recall that a Frechet space is a metrizable locally convex complete TVS, alternatively, it is a locally convex complete TVS whose topology is induced by a countable family of seminorms.

stable is the ∞-analogue of being an abelian category.) By gluing, we obtain a category C_X for any complex analytic space X. It can be viewed as the derived (∞-)category of *quasicoherent* liquid sheaves on X.

This is cool! Because until now, there was only a good theory of *coherent* sheaves on analytic spaces. Of course we have to justify that these categories deserve to be called the derived categories of quasicoherent liquid sheaves.

Well, it doesn't stop with the definition of these categories. Another missing piece of the puzzle, that can now be filled in, is the exceptional pushforward functor. This functor should be part of a six functor formalism. We will sketch some parts of it in the remainder of this talk.

7. Six functors

7.1. For any morphism $f\colon X \to Y$ of complex analytic spaces,[2] there is a natural functor $f^*\colon C_Y \to C_X$ that preserves all colimits. By abstract nonsense, this means that it admits a right adjoint $f_*\colon C_X \to C_Y$.

7.2. The categories C_X are naturally closed symmetric monoidal. This means that they have an internal Hom, and a symmetric tensor product, which are adjoint in the expected way:

$$\mathrm{Hom}(M \otimes N, P) \cong \mathrm{Hom}(M, \underline{\mathrm{Hom}}(N, P))$$

7.3. Let X be a Hausdorff complex analytic space, and let $\mathcal{F} \in C_X$. The set of open subsets U for which $\mathcal{F}|_U = 0$ is closed under arbitrary unions, by the sheaf property. Hence there is a maximal such U, and its complement is the *support* of $\mathcal{F}$, denoted by $\operatorname{Supp}\mathcal{F}$. If $\operatorname{Supp}\mathcal{F}$ is compact, we say that $\mathcal{F}$ is *compactly supported*.

Theorem 7.4 ([1, Thm 12.15])**.** *Let $f\colon X \to Y$ be a morphism of Hausdorff complex analytic spaces. Then there exists a unique colimit-preserving functor*

$$f_!\colon C_X \to C_Y$$

equipped with a natural isomorphism of $f_!$ with f_* when restricted to the full subcategory of compactly supported objects of C_X.

Fact 7.5. The functor $f_!\colon C_X \to C_Y$ admits a right adjoint

$$f^!\colon C_Y \to C_X.$$

(Up to some size issues, that can be dealt with in various ways. For example, by going pyknotic...)

7.6. The six functors

$$f^*, f_*, f_!, f^*, \underline{\mathrm{Hom}}, \otimes$$

satisfy a list of expected formal compatibilities that mostly roll out of the formalism. Here are some of them:

- Functorialities: $(g\circ f)^* = f^* \circ g^*$, $(g\circ f)_* = g_* \circ f_*$, $(g\circ f)_! = g_! \circ f_!$, and $(g\circ f)^! = f^! \circ g^!$.
- If f is proper, then $f_! = f_*$.
- If f is an open immersion, then $f_!$ is left adjoint to $f^!$.
- There is a projection formula, $\mathcal{G} \otimes f_!\mathcal{F} \cong f_!((f^*\mathcal{G}) \otimes \mathcal{F})$ for $\mathcal{G} \in C_Y$ and $\mathcal{F} \in C_X$.

[2]NB: in [1], the notion of complex analytic space has been *generalized* to include objects with boundary, for example. We will ignore that generalization.

- There is a base change formula. Given a pullback square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \Big\downarrow{\scriptstyle g'} & & \Big\downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

there is a natural isomorphism $f^* \circ g_! \cong (g')_! \circ (f')^*$.

(We blatantly ignore all coherence issues between these isomorphisms, etc...)

Theorem 7.7 (Serre duality, [1, Thm 13.6])**.** *Let $f\colon X \to Y$ be a smooth morphism of dimension d between complex analytic spaces. Then there is a natural isomorphism*

$$f^! M \cong f^* M \otimes_{\mathcal{O}_X} \Omega^d_{X/Y}$$

for $M \in C_Y$.

7.8. The proof consists of two components:

(1) The computation that $f^! \mathcal{O}_Y = \Omega^d_{X/Y}[d]$. This is done by deformation to the normal cone. The argument fits on one page.
(2) Abstract manipulations in the six functor formalism, using formal properties of the type listed above.

7.9. Several other fundamental results in complex geometry can have their proofs simplified by using the liquid machinery. In "Complex Geometry" [1], Clausen and Scholze reprove: but also

- Serre duality (as we saw above). Note that it is generalized from coherent sheaves to quasicoherent sheaves.
- GAGA. Again, in the quasicoherent setting, generalizing the coherent case.
- Finiteness of coherent cohomology.
- Hirzebruch–Riemann–Roch. (I admit that I still find this proof intimidating and involved.)

Clausen also showed that the comparison isomorphism between algebraic and analytic de Rham cohomology can be established by formally reducing to the 1-dimensional cases of the disk and the punctured disk.

References

1. Dustin Clausen and Peter Scholze, *Condensed Mathematics and Complex Geometry*, (2022), `https://people.mpim-bonn.mpg.de/scholze/Complex.pdf`.
2. Peter Scholze, *Lecture notes on Condensed Mathematics*, (2019), `https://www.math.uni-bonn.de/people/scholze/Condensed.pdf`.
3. ———, *Lectures on Analytic Geometry*, (2019), `https://www.math.uni-bonn.de/people/scholze/Analytic.pdf`.
4. Lucien Waelbroeck, *Topological vector spaces and algebras*, Lecture Notes in Mathematics, Vol. 230, Springer-Verlag, Berlin-New York, 1971. MR 0467234