On the Mumford–Tate conjecture for surfaces with $p_g = q = 2$

by J.M. Commelin 2015–10–27

0.1 Abstract. — The Mumford-Tate conjecture is a precise way to say that two invariants of an algebraic variety X (over a number field) convey the same information. The two invariants in question are (1) the Hodge structure on the singular cohomology of the complex analytic variety associated with X; and (2) the Galois representation on the ℓ-adic etale cohomology of X. The conjecture fits in a bigger framework of conjectures, like the Hodge conjecture and the Tate conjecture; but the factual evidence is very small. In this talk I will present joint work with Matteo Bonfanti and Matteo Penegini, on the Mumford-Tate conjecture for surfaces with $p_g = q = 2$.

1 The Mumford–Tate conjecture

Let k be a number field: a finite field extension of $\mathbb{Q}$. (If you want, just take $k = \mathbb{Q}$.) Fix an embedding $k \subset \mathbb{C}$. Let X be a connected smooth projective variety over k. You may think of this as the solution set in $\mathbb{P}^n(k)$ of a set of homogeneous polynomial equations with coefficients in k. The smoothness property has to do with some Jacobian criterion, like in calculus.

1.1 Hodge theory. — The complex points $X(\mathbb{C})$ form a smooth projective complex manifold. The cohomology $\mathrm{H}^i_{\mathrm{B}}(X) = \mathrm{H}^i(X(\mathbb{C}), \mathbb{Q})$ caries a Hodge structure, by a deep theorem of Hodge.

We give two definitions of Hodge structures.

1.2 Definition. — A *Hodge structure* is a finite-dimensional $\mathbb{Q}$-vector space V, together with a decomposition $V \otimes \mathbb{C} \cong \bigoplus_{p,q \in \mathbb{Z}} V^{p,q}$ such that $V^{p,q} \cong \overline{V^{q,p}}$. «

1.3 Definition. — A *Hodge structure* is a finite-dimensional $\mathbb{Q}$-vector space V, together with a representation $\mathbb{S} \to \mathrm{GL}(V \otimes \mathbb{R})$, where $\mathbb{S}$ is the 2×2-matrix representation of $\mathbb{C}^*$ over $\mathbb{R}$. «

Roughly speaking, the equivalence between these is the following: an element $z \in \mathbb{C}^*$ acts on $V^{p,q}$ as multiplication by $z^{-p}\bar{z}^{-q}$. (The minus signs are a convention that history forces upon us.)

1.4 Definition. — The *Mumford–Tate group* of a Hodge structure V is the smallest algebraic subgroup $\mathrm{G_B} \subset \mathrm{GL}(V)$ such that $\mathbb{S} \to \mathrm{GL}(V \otimes \mathbb{R})$ factors through $\mathrm{G_B} \otimes \mathbb{R} \subset \mathrm{GL}(V \otimes \mathbb{R})$. «

We will get to interesting facts about the Mumford–Tate group in a minute.

1.5 Étale cohomology. — There is another cohomology theory for algebraic varieties, that can even be applied in positive characteristic: ℓ-adic étale cohomology. This cohomology theory spits out vector spaces over the field $\mathbb{Q}_\ell$, the field of ℓ-adic numbers. This is a completion of $\mathbb{Q}$ for a "non-standard" norm, depending on a prime ℓ. In particular it contains $\mathbb{Q}$. Let us write $\mathrm{H}^i_\ell(X)$ for these cohomology groups.

There are two very cool things about $\mathrm{H}^i_\ell(X)$.

(a) There is a natural action of a group on it: $\Gamma_k = \mathrm{Gal}(\bar{k}/k) = \mathrm{Aut}_k(\bar{k})$. So we have a homomorphism $\Gamma_k \to \mathrm{GL}(\mathrm{H}^i_\ell(X))$.
(b) A deep theorem of Artin says that we can compare $\mathrm{H}^i_\ell(X)$ with $\mathrm{H}^i_{\mathrm{B}}(X)$: there is a natural isomorphism $\mathrm{H}^i_{\mathrm{B}}(X) \otimes \mathbb{Q}_\ell \cong \mathrm{H}^i_\ell(X)$.

Let G_ℓ be the smallest algebraic subgroup of $GL(H^i_\ell(X))$ that contains the image of Γ_k. This group does not need to be connected, but the group of connected components is finite.

1.6 Definition. — The group G_ℓ is called the *ℓ-adic algebraic monodromy group* of $H^i_\ell(X)$. We write G°_ℓ for the connected component of the identity of G_ℓ. «

1.7 Big motivic conjectures. — To give a quick hint why these representations should be useful, consider the following two conjectures.

- The Hodge conjecture says that invariants in $H_B(X)$ under the Mumford–Tate group are precisely the classes of closed algebraic subvarieties.
- The Tate conjecture says that invariants in $H_\ell(X)$ under the Galois group are precisely the classes of closed algebraic subvarieties.

If you solve the first one, you get a Fields medal, and 1 million dollars. If you solve the second one, you only get a Fields medal.

1.8 The Mumford–Tate conjecture. — The Mumford–Tate conjecture $MTC(X)$ says that under the comparison isomorphism of Artin: $G_B \otimes \mathbb{Q}_\ell \cong G^\circ_\ell$

So now we have three conjectures:

(a) The Hodge conjecture for all varieties X over k.
(b) The Tate conjecture for all varieties X over k.
(c) The Mumford–Tate conjecture for all varieties X over k.

If any two of these are known, the third follows.

All these conjectures are only known for (very) special cases of X. For the Mumford–Tate conjecture, the list is not much longer then this:

- abelian varieties of dimension $g \leq 3$;
- curves of genus $g \leq 3$;
- K3 surfaces and some other surfaces with $p_g = 1$.

There are more examples, but most of them require special conditions, and would take way too long to spell out here. One very important result towards the Mumford–Tate conjecture is by Deligne. He proved that for abelian varieties we have $G^\circ_\ell \subset G_B \otimes \mathbb{Q}_\ell$.

2 Surfaces with $p_g = q = 2$

Assume that X is a surface: the dimension of X is 2. One can associate invariants p_g and q with X, and almost by definition, you can read them off from the *Hodge diamond* of X, which looks like:

$$\begin{matrix} & & 1 & & \\ & q & & q & \\ p_g & & h^{11} & & p_g \\ & q & & q & \\ & & 1 & & \end{matrix}$$

For example, for an abelian surface, we have $p_g = 1$, and $q = 2$; while for a K3 surface, we have $p_g = 1$, and $q = 0$.

For this talk, we are also interested in surfaces with $p_g = q = 2$. Every surface has a “universal” map to an abelian variety of dimension q, called the *Albanese morphism*, to the *Albanese variety* of X. For the purpose of this talk, we want this Albanese morphism to be surjective (which is possible, if $q = 2$). The current state of the art in describing the moduli space of such surfaces is the following: we know there are at least 12 connected components of the moduli space, and all of these have dimension at least 2. Also, under a certain minimality condition, all such surfaces have invariant $h^{11} \leq 10$.

3 Motives

Let $X \to A$ be the Albanese morphism. Since we assume it is surjective, we get injective maps $\mathrm{H_B}(A) \to \mathrm{H_B}(X)$ and $\mathrm{H}_\ell(A) \to \mathrm{H}_\ell(X)$. Since A is an abelian surface, it has invariants $p_g = 1$ and $q = 2$. We see that $\mathrm{H_B}(X) = \mathrm{H_B}(X)_{\mathrm{old}} \oplus \mathrm{H_B}(X)_{\mathrm{new}}$ and $\mathrm{H}_\ell(X) = \mathrm{H}_\ell(X)_{\mathrm{old}} \oplus \mathrm{H}_\ell(X)_{\mathrm{new}}$, where $\mathrm{H_B}(X)_{\mathrm{old}} \cong \mathrm{H_B}(A)$ and $\mathrm{H}_\ell(X)_{\mathrm{old}} \cong \mathrm{H}_\ell(A)$. The summands $\mathrm{H_B}(X)_{\mathrm{new}}$ and $\mathrm{H}_\ell(X)_{\mathrm{new}}$ form an example of a so called *motive*, which we will denote with $\mathrm{H}(X)_{\mathrm{new}}$. We call $\mathrm{H_B}(X)_{\mathrm{new}}$ and $\mathrm{H}_\ell(X)_{\mathrm{new}}$ the realisations of $\mathrm{H}(X)_{\mathrm{new}}$.

Roughly speaking, a motive is a summand of the cohomology of an algebraic variety that is cut out by morphisms of algebraic varieties (or more general correspondences). There are lots of results about motives, but there are also still a lot of foundational questions; so in some sense there is not even a satisfactory definition of motives yet. However, we can not go into that now.

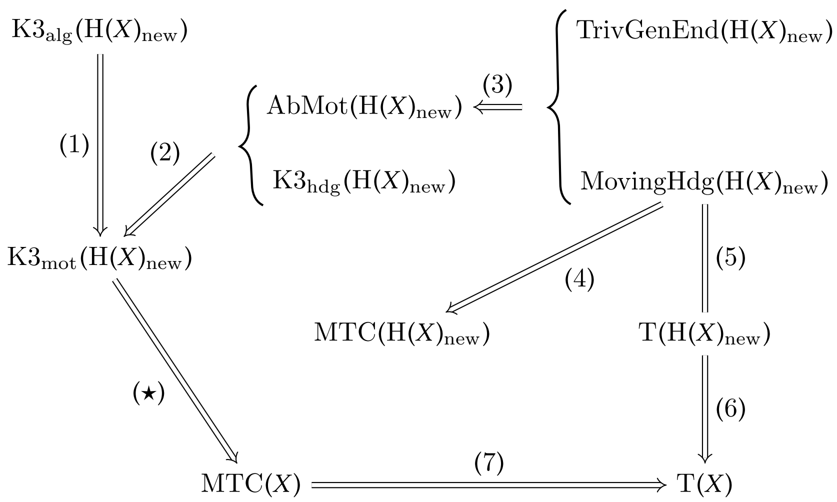
We do note the following observations:

» If the motive is an abelian motive (that is, a summand of the cohomology of an abelian variety), then Deligne's theorem $\mathrm{G}_\ell^\circ \subset \mathrm{G_B} \otimes \mathbb{Q}_\ell$ is still true.
» For abelian motives, morphisms of motives are the same as morphisms of their Hodge realisations.
» There is a natural generalisation of the Mumford–Tate conjecture to motives.
» If M_1 and M_2 are two motives, and $\mathrm{MTC}(M_1)$ and $\mathrm{MTC}(M_2)$ are known, then $\mathrm{MTC}(M_1 \oplus M_2)$ may still be a very hard problem.
» The motive $\mathrm{H}(X)_{\mathrm{new}}$ has "invariant" $p_g = 1$.

4 The main result

We want to prove $\mathrm{MTC}(X)$.

4.1 A bunch of conjectures. — We now give a big graph of conjectures, and implications between them. Afterwards, we explain these conjectures and implications.



$\mathrm{K3}_{\mathrm{hdg}}(M)$	There exists a K3 surface Y, and an injective morphism of Hodge structures $\mathrm{H}_{\mathrm{B}}(M) \to \mathrm{H}^2_{\mathrm{B}}(Y)$.
$\mathrm{K3}_{\mathrm{mot}}(M)$	There exists a K3 surface Y, and an injective morphism of motives $M \to \mathrm{H}(Y)$.
$\mathrm{K3}_{\mathrm{alg}}(\mathrm{H}(X)_{\mathrm{new}})$	There exists a K3 surface Y, and an algebraic correspondence in $X \times Y$, inducing an injective morphism of motives $\mathrm{H}(X)_{\mathrm{new}}) \to \mathrm{H}(Y)$.
$\mathrm{MovingHdg}(M)$	The motive M can be put as a fibre into a family of motives $\mathcal{M}/B$ over some base B, such that the variation of Hodge structure (corresponding to the Hodge realisation) is not constant.
$\mathrm{TrivGenEnd}(M)$	The motive M can be put as a fibre into a family of motives $\mathcal{M}/B$ over some base B, such that the generic endomorphism ring of the family is $\mathbb{Q}$.
$\mathrm{AbMot}(M)$	The motive M is an abelian motive.
$\mathrm{MTC}(M)$	The Mumford–Tate conjecture is true for the motive M.
$\mathrm{T}(M)$	The Tate conjecture is true for the motive M.

The implication $(\star)$ is not officialy proven, but I am working on a proof. (All the cases we need or done, but I am working on the general case.) Let me justify all other implications.

(1) Every algebraic cycle is a motivated cycle. So if there is an algebraic cycle inducing an injective morphism of Hodge structures $H^2(S)_{\mathrm{new}} \cong H^2(X)$, then there certainly exists a motivated cycle doing the job.
(2) For this implication, we are given that there is a Hodge cycle inducing an injective morphism of Hodge structures $\mathrm{H}(X)_{\mathrm{new}} \to \mathrm{H}(Y)$, and we are also given that the motive $\mathrm{H}(X)_{\mathrm{new}}$ is an abelian motive. By théorème 0.6.3 of [1], the motive of the K3 surface Y is an abelian motive. Now we use the remark that morphisms between abelian motives are the same as morphisms between their Hodge realisations.
(3) This is buried in the paper of Ben Moonen, and also in some paper of Yves André.
(4) This is the main theorem of [2].
(5) This is also the main theorem of [2].
(6) The Tate conjecture is additive on motives. It says that Galois invariant cycles are algebraic cycles. If this is true for two motives M_1 and M_2, then indeed, it is also true for $M_1 \oplus M_2$; because the subspace of Galois invariants of $\mathrm{H}_\ell(M_1 \oplus M_2)$ is just the sum of the Galois invariants of $\mathrm{H}_\ell(M_1)$ and the Galois invariants of $\mathrm{H}_\ell(M_2)$.
(7) Since the Hodge conjecture is true for X (because it is a surface, Lefschetz (1,1) theorem), the Mumford–Tate conjecture implies the Tate conjecture.

Please observe that there is no arrow from $\mathrm{MTC}(\mathrm{H}(X)_{\mathrm{new}})$ to $\mathrm{MTC}(\mathrm{H}(X))$, since the Mumford–Tate conjecture is *not* additive, contrary to the Tate and Hodge conjectures.

4.2 Application to our surfaces. — From the graph of implications, it is clear that we are done if we prove

» $\mathrm{K3}_{\mathrm{hdg}}(\mathrm{H}(X)_{\mathrm{new}})$, or
» $\mathrm{MovingHdg}(\mathrm{H}(X)_{\mathrm{new}})$ and $\mathrm{TrivGenEnd}(\mathrm{H}(X)_{\mathrm{new}})$.

We gave a short description of what is known about the moduli space of surfaces with $p_g = q = 2$ and surjective Albanese morphism. Of the 12 known components, there are 7 components that contain so called product-quotients. That means that some of the surfaces in the component are of the following form: there is a group G, and two curves C_1 and C_2, both with an action of G, so that X is a resolution of singularities of $(C_1 \times C_2)/G$. (Here the action of G on $C_1 \times C_2$ is the diagonal action.)

All such surfaces with $p_g = q = 2$ have been classified by my coauthor M. Penegini. We can prove $\mathrm{K3}_{\mathrm{hdg}}(\mathrm{H}(X)_{\mathrm{new}})$ for such surfaces. We use a theorem of Chevalley and Weil to understand the representation of G on $\mathrm{H}_{\mathrm{B}}(C_i)$, which gives us grip on the action of G on the Jacobian $J(C_i)$. This helps us to find an abelian surface B as subvariety of $J(C_1) \times J(C_2)$, such that $\mathrm{H}(X)_{\mathrm{new}}$ corresponds with $\mathrm{H}^2(B)$ via an algebraic correspondence. The K3 surface Y that we are looking for is then the Kummer variety of B. (That is, a resolution of singularities of $B/\langle -1 \rangle$.)

To get a grip on the other surfaces in these 7 components, we still zoom in on this locus of product-quotients. Using a lemma of Pirola, we check that the Hodge structure $\mathrm{H}^2_{\mathrm{B}}(X)_{\mathrm{new}}$ changes, if we deform (G, C_1, C_2). In other words, if we change X a bit, into another product-quotient, then the Hodge structure also changes. (This does not work for all product quotients, but in each component, we can find product-quotients for which it works.)

Now let X be any surface in one of these components. Let X_1 and X_2 be two product-quotients in that same component of the moduli space. We may now find a curve C in this component of the moduli space, that passes through X, X_1, and X_2. This shows that $\mathrm{H}(X)_{\mathrm{new}}$ may be placed in a family (over C) for which the Hodge realisation is non-constant.

Now we only need to prove that the generic endomorphism ring if trivial. Since the dimension of $\mathrm{H}_{\mathrm{B}}(X)_{\mathrm{new}}$ is quite small, there are not so many options, and by looking at the possible endomorphism rings of such Hodge structures, we see that it must be $\mathbb{Q}$, or a field of degree 2 over $\mathbb{Q}$, if the Hodge structure moves. By looking at the product-quotients again, we can prove that it must be $\mathbb{Q}$. This proves the Mumford–Tate conjecture for surfaces in these 7 components.

We have some ideas (and good hope) to attack the other components, but I will not go into that now.

Bibliography

[1] Yves André. "Pour une théorie inconditionnelle des motifs". In: *Inst. Hautes Études Sci. Publ. Math.* 83 (1996), pp. 5–49.

[2] Ben J. J. Moonen. *On the Tate and Mumford-Tate conjectures in codimension one for varieties with $h^{2,0} = 1$.* URL: http://arxiv.org/abs/1504.05406.