Finite group schemes

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Contents

1 References

- The main reference is §3 of the manuscript of Moonen and his coauthors.
- For some useful facts on connected (resp. reduced) schemes, see EGA IV.
- If you are hardcore, the most general version of any statement about group schemes can be found in SGA3.

2 Examples

2.1 Examples we have seen before

Let S be a scheme. We recall some examples of group schemes you have already seen.

- The group scheme $\mathbb{G}\mathrm{a}_S$ is defined by the functor

$$\begin{aligned} \mathbb{G}\mathrm{a}_S \colon \mathrm{Sch}_{/S}^{\mathrm{op}} &\longrightarrow \mathrm{Grp} \\ T &\longmapsto (\mathcal{O}_T(T), +) \end{aligned}$$

  It is represented by the scheme $\mathbb{A}^1 \times S = \mathrm{Spec}(\mathbb{Z}[X]) \times S$. If S is affine, say $\mathrm{Spec}(A)$, then $\mathbb{G}\mathrm{a}_S \cong \mathrm{Spec}(A[X])$.

- The group scheme $\mathbb{G}\mathrm{m}_S$ is defined by the functor

$$\begin{aligned} \mathbb{G}\mathrm{m}_S \colon \mathrm{Sch}_{/S}^{\mathrm{op}} &\longrightarrow \mathrm{Grp} \\ T &\longmapsto \mathcal{O}_T(T)^* \end{aligned}$$

  It is represented by the scheme $\mathbb{G}\mathrm{m}_{\mathbb{Z}} \times S = \mathrm{Spec}(\mathbb{Z}[X, X^{-1}]) \times S$. If S is affine, say $\mathrm{Spec}(A)$, then $\mathbb{G}\mathrm{m}_S \cong \mathrm{Spec}(A[X, X^{-1}])$.

If $S' \to S$ is a morphism of schemes, then $\mathbb{G}\mathrm{a}_{S'} \cong \mathbb{G}\mathrm{a}_S \times_S S'$ and $\mathbb{G}\mathrm{m}_{S'} \cong \mathbb{G}\mathrm{m}_S \times_S S'$. This is immediate from the way we gave the representing schemes in the above examples.

These examples naturally lead to the definition of the following subgroup schemes.

- The subgroup scheme $\mu_{n,S} \subset \mathbb{G}\mathrm{m}_S$ is defined by the functor

$$\begin{aligned} \mu_{n,S} \colon \mathrm{Sch}_{/S}^{\mathrm{op}} &\longrightarrow \mathrm{Grp} \\ T &\longmapsto \{x \in \mathcal{O}_T(T)^* \mid x^n = 1\} \end{aligned}$$

  It is represented by $\mathrm{Spec}(\mathbb{Z}[X]/(X^n - 1)) \times S$.

- Assume the characteristic of S is a prime $p > 0$. (In other words, $\mathcal{O}_S(S)$ is a ring of characteristic p; or equivalently, $S \to \mathrm{Spec}(\mathbb{Z})$ factors via $\mathrm{Spec}(\mathbb{F}_p)$.) The subgroup scheme $\alpha_{p^n,S} \subset \mathbb{G}\mathrm{a}_S$ is defined by the functor

$$\begin{aligned} \alpha_{p^n,S} \colon \mathrm{Sch}_{/S}^{\mathrm{op}} &\longrightarrow \mathrm{Grp} \\ T &\longmapsto \{x \in \mathcal{O}_T(T)^* \mid x^{p^n} = 0\} \end{aligned}$$

  It is represented by $\mathrm{Spec}(\mathbb{Z}[X]/(X^p)) \times S$.

In a moment we will see that $\mu_{n,S}$ and $\alpha_{p^n,S}$ are examples of kernels.

Example 1 Observe that if we forget the group structures, then $\mu_{p^n,S}$ and $\alpha_{p^n,S}$ represent the same functor. Indeed, they are fibres of the same homomorphism of rings. However, as group schemes they are not isomorphic.

2.2 Constant group schemes

Let G be an abstract group. We associate a group scheme with G, the so called *constant group scheme* G_S. It is defined by the functor

$$G_S \colon \operatorname{Sch}_{/S}^{\mathrm{op}} \longrightarrow \operatorname{Grp}$$
$$T \longmapsto G^{\pi_0(T)}$$

It is represented by $\coprod_{g\in G} S$. Indeed, if T is connected,

$$\operatorname{Hom}_S(T, \coprod_{g\in G} S) = G_S(T)$$

because T must be mapped to exactly one copy of S, and the mapping must be the structure morphism $T \to S$. For general T, the identity follows from abstract nonsense:

$$\operatorname{Hom}(\coprod_{i\in I} T_i, X) = \prod_{i\in I} \operatorname{Hom}(T_i, X)$$

Example 2 Let k be a field of characteristic p. Let n be an integer that is not divisible by p. In general $(\mathbb{Z}/n\mathbb{Z})_k$ and $\mu_{n,k}$ are not isomorphic. However, if k contains a primitive n-th root of unity (for example if k is algebraically closed), then $(\mathbb{Z}/n\mathbb{Z})_k \cong \mu_{n,k}$.

We say that μ_n is a *form* of the constant group scheme $(\mathbb{Z}/n\mathbb{Z})_k$. Later on we hope to see that, if k is a field of characteristic 0, then every finite group scheme over k is a form of a constant group scheme. Moreover, if k is algebraically closed, then every finite group scheme is constant.

2.3 Kernel of a homomorpism of group schemes

Let $f\colon G \to H$ be a homomorphism of group schemes over some scheme S. The kernel subgroup scheme $\operatorname{Ker}(f) \subset G$ is defined via the functor

$$\operatorname{Ker}(f)\colon \operatorname{Sch}_{/S}^{\mathrm{op}} \longrightarrow \operatorname{Grp}$$
$$T \longmapsto \operatorname{Ker}(G(T) \to H(T))$$

This functor is representable, because it is a pullback

$$\begin{array}{ccc} \operatorname{Ker}(f) & \longrightarrow & G \\ \downarrow & & \downarrow f \\ S & \xrightarrow{\;1\;} & H \end{array}$$

Note that μ_n is the kernel

$$[n]\colon \mathbb{G}\mathrm{m} \longrightarrow \mathbb{G}\mathrm{m}$$
$$x \longmapsto x^n$$

and similarly α_{p^n} is the kernel of Frobenius

$$\operatorname{Frob}_p\colon \mathbb{G}\mathrm{a} \longrightarrow \mathbb{G}\mathrm{a}$$
$$x \longmapsto x^{p^n}$$

2.4 Multiplication by n

Let S be a scheme. Let G/S be a commutative group scheme over S. For every non-negative integer $n \in \mathbb{Z}_{\geq 0}$ there is a group scheme homomorphism "multiplication by n" given by

$$[n]\colon\ G \longrightarrow G$$
$$x \longmapsto n \cdot x$$

(Here we use additive notation for G.)

The kernel of this morphism is usually denoted $G[n]$.

Note that we can define μ_n as $\mathbb{G}\mathrm{m}[n]$.

2.5 Semidirect product of group schemes

Let N and Q be two group schemes over a basis S. Let

$$\mathbf{Aut}(N)\colon\ \mathrm{Sch}_{/S}^{\mathrm{op}} \longrightarrow \mathrm{Grp}$$
$$T \longmapsto \mathrm{Aut}(N_T)$$

denote the automorphism functor of N. (By the way, with $\mathrm{Aut}(N_T)$ we mean automorphisms of N_T *as group scheme*!) Let $\rho\colon Q \to \mathbf{Aut}(N)$ be an action of Q on N.

The *semi-direct product group scheme* $N \rtimes_\rho Q$ is defined by the functor

$$N \rtimes_\rho Q\colon\ \mathrm{Sch}_{/S}^{\mathrm{op}} \longrightarrow \mathrm{Grp}$$
$$T \longmapsto N(T) \rtimes_{\rho_T} Q(T)$$

which is represented by $N \times_S Q$. Recall that if (n, q) and (n', q') are T-valued points of $N \rtimes_\rho Q$, then

$$(n, q) \cdot (n', q') = (n \cdot \rho(q)(n'), q \cdot q').$$

3 Étale schemes over fields

3.1 Étale morphisms

We now give two definition of étale morphisms; but we do not show that they are equivalent.

Definition 1 A morphism of schemes $X \to S$ is *étale* if it is flat and unramified.

Observe that

- $X \to \mathrm{Spec}(k)$ is always flat (trivial);
- $X \to \mathrm{Spec}(k)$ is unramified if it is locally of finite type and if for all $x \in X$ the ring map $k \to \mathcal{O}_{X,x}$ is a finite separable field extension.

Definition 2 A morphism of schemes $X \to S$ is *formally étale* if for every

- commutative ring A,
- and every ideal $I \subset A$, such that $I^2 = 0$,

- and every commutative square

$$\begin{array}{ccc} \mathrm{Spec}(A/I) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & S \end{array}$$

there exists precisely one map $\mathrm{Spec}(A) \to X$ such that

$$\begin{array}{ccc} \mathrm{Spec}(A/I) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & S \end{array}$$

commutes.

Proposition 1 *A morphism of schemes $X \to S$ is étale if and only if it is locally of finite presentation and formally étale.*

Example 3 In other words, a group scheme G/k over a field k is étale if for every k-algebra A, and every ideal $I \subset A$ with $I^2 = 0$, the map $G(A) \to G(A/I)$ is a bijection.

We now specialise to the case $S = \mathrm{Spec}(k)$, with k a field. Fix a separable closure $\bar{k}$ of k.

Theorem 1 *The functor*

$$\begin{array}{rcl} \{\textit{ét. sch. over } k\} & \longrightarrow & \{\textit{disc. ctu. } \mathrm{Gal}(\bar{k}/k)\textit{-sets}\} \\ X & \longmapsto & X(\bar{k}) \end{array}$$

is an equivalence of categories.

Proof Every discrete $\mathrm{Gal}(\bar{k}/k)$-set is a disjoint union of orbits. Every orbit is stabilised by a finite index subgroup $H \subset \mathrm{Gal}(\bar{k}/k)$. The orbit corresponds to $\mathrm{Spec}(\bar{k}^H)$.

Conversely, every étale scheme over k is the disjoint union of its connected components; and every connected étale scheme over k is a field extension.

3.2 Étale group schemes over fields

The theorem allows us to describe étale group schemes over k as group objects in the category of discrete $\mathrm{Gal}(\bar{k}/k)$-sets. In other words, a étale group scheme G/k is fully described by

- the group $G(\bar{k})$, together with
- the action of $\mathrm{Gal}(\bar{k}/k)$ on $G(\bar{k})$.

Vice versa, every group discrete G together with a continuous action of $\mathrm{Gal}(\bar{k}/k)$ acting via automorphisms of G (or equivalently, such that the multiplication $G \times G \to G$ is Galois equivariant) determines a étale group scheme over k.

4 Standard constructions

Let G be a finite (hence affine) k-group scheme. By the *rank* of G we mean the k-dimension of its affine algebra $\mathcal{O}_G(G)$. For example, $\mu_{p,k}$, $\alpha_{p,k}$ and $(\mathbb{Z}/p\mathbb{Z})_k$ all have rank p.

4.1 Connected component of the identity

Let G/k be a group scheme over some field k. Let G^0 denote the connected component of G that contains e. One expects that G^0 is a subgroup scheme of G. This is indeed true. One needs to prove that the image of $G^0 \times_k G^0 \subset G \times_k G$ under the multiplication map $m\colon G \times_k G \to G$ is contained in G^0.

We are done if $G^0 \times_k G^0$ is connected.

In general, if $X \to S$ and $Y \to S$ are S-schemes, and X and Y are connected, then $X \times_S Y$ need not be connected. For example take $\mathbb{C}/\mathbb{R}$ for X/S and Y/S.

However, we have a rational point $e \in G^0(k)$ at our disposal.

Lemma 1 *Let X/k be a k-scheme that is locally of finite type. Assume X is connected and has a rational point $x \in X(k)$. Then X is geometrically connected.*

Proof Let L/k be a field extension. It suffices to show that the projection $p\colon X_L \to X$ is open and closed. The properties of being open and closed are local on the target. In other words, if $(U_i)_{i\in I}$ is an affine cover of X, then $(p^{-1}(U_i))_{i\in I}$ covers X_L, and if every $p^{-1}(U_i) \to U_i$ is open and closed, then so is p. Note that $p^{-1}(U_i) = U_{i,L}$.

Hence we may assume that X is affine and of finite type. Let $Z \subset X_L$ be closed. Then there exists a field K, with $k \subset K \subset L$, and K/k finite, such that Z is defined over K. Concretely, there exists a $Z' \subset X_K$, such that $(Z')_L = Z$.

Thus, for every closed (and therefore, for every open) subset of X we have reduced the question to whether $X_K \to X$ is open and closed for finite extension K/k. But K/k is finite and flat, hence so is $X_K \to X$. But finite flat morphisms are open and closed (use HAG, Chap. III, Ex. 9.1 or EGA IV, Thm. 2.4.6.).

The lemma shows that G^0 is geometrically connected. This implies that $(G^0)_K = (G_K)^0$ for every field extension K/k.

Moreover, $G^0 \times_k G^0$ is connected, by `http://stacks.math.columbia.edu/tag/0385`. It follows that G^0 carries a subgroup scheme structure.

Together, we have proved parts of the following theorem.

Theorem 2 (Parts of proposition 3.17 from the manuscript) *Let G be a group scheme, locally of finite type over a field k.*

(i) The identity component G^0 is an open and closed subgroup scheme of G that is geometrically irreducible. In particular: for any field extension $k \subset K$, we have $(G^0)_K = (G_K)^0$.

(ii) The following properties are equivalent:

(a1) $G \times_k K$ is reduced for some perfect field K containing k;

(a2) the ring $\mathcal{O}_{G,e} \otimes_k K$ is reduced for some perfect field K containing k;

(b1) G is smooth over k;

(b2) G^0 is smooth over k;

(b3) G is smooth over k at the origin.

PROOF The lemma gives us most of (i).

The flavour for most of (ii) can be grabbed from `http://stacks.math.columbia.edu/tag/04QM`. Indeed (a1) $\implies$ (a2) and (b1) $\implies$ (b2) $\implies$ (b3) are trivial.

Example 4 (i) Let k be a non-perfect field. Let $\alpha \in k$ be an element that is not a p-th power. Observe that $G = \mathrm{Spec}(k[X,Y]/(X^p + \alpha Y^p)$ is a closed subgroup scheme of $\mathbb{A}^2_k$. It is reduced, but not geometrically reduced, hence not smooth. (ii) Consider $\mu_{n,\mathbb{Q}}$, for $n > 2$. The connected component of the identity is geometrically irreducible (as the theorem says) but all other components split into more components after extending to $\bar{\mathbb{Q}}$.

4.2 Component scheme

Let k be a field. Let X/k be a scheme, locally of finite type.

The inclusion functor

$$\{\text{ét } k\text{-schemes}\} \longrightarrow \{\text{loc. fin. type } \mathrm{Sch}_{/k}\}$$

admits a left adjoint

$$\varpi_0 \colon \{\text{loc. fin. type } \mathrm{Sch}_{/k}\} \longrightarrow \{\text{ét } k\text{-schemes}\}$$

In other words, every morphism $X \to Y$ of k-schemes, with Y/k étale, factors uniquely via $X \to \varpi_0(X)$.

To understand what $\varpi_0(X)$ is, we use our description of étale k-schemes.

Fix a separable closure $\bar{k}/k$. Observe that $\mathrm{Gal}(\bar{k}/k)$ acts on $\mathrm{Spec}(\bar{k})$, hence on, $X_{\bar{k}} = X \times_k \mathrm{Spec}(\bar{k})$, hence on the topological space underlying X_k, hence on $\pi_0(X_{\bar{k}})$.

The claim is then, that this action is continuous. Indeed, every connected component $C \in \pi_0(X_{\bar{k}})$ is defined over some finite extension $k' \subset \bar{k}$ of k, and therefore the stabiliser of C contains the open subgroup $\mathrm{Gal}(\bar{k}/k')$. (See the manuscript §3.27 for details.) The étale k-scheme associated with this action is $\varpi_0(X)$.

This shows that ϖ_0 is a functor, as claimed. It is the identity on étale k-schemes. Consequently, every map $X \to Y$ to an étale scheme induces a map $\varpi_0(X) \to Y$.

There is an obvious map $X_{\bar{k}} \to \varpi_0(X_{\bar{k}})$. This map is $\mathrm{Gal}(\bar{k}/k)$-equivariant, and therefore we get a map $X \to \varpi_0(X)$. The fibers of this map are precisely the connected components of X (as open subschemes of X).

4.2.1 Component group

Let G/k be a group scheme, locally of finite type. Since $G^0 \subset G$ is a normal subgroup scheme, there is a natural group scheme structure on $\varpi_0(G)$. In particulare we get the following short exact sequence of group schemes.

$$1 \mapsto G^0 \to G \to \varpi_0(G) \to 1$$

4.3 Reduced group scheme

Let k be a field. Let G/k be a group scheme. Let G_{red} be the underlying reduced scheme of G.

It is natural to ask if G_{red} is carries a natural group scheme structure over k. In general the answer is no.

However, if we assume k is perfect, the answer is yes. Since G_{red} is reduced, it is smooth (the theorem on connected components), and therefore geometrically reduced (again the theorem). By EGA IV 4.6.1, this implies that $G_{\mathrm{red}} \times_k G_{\mathrm{red}}$ is reduced, and therefore is mapped to G_{red} under the multiplication map $G \times_k G \to G$.

In general $G_{\mathrm{red}} \subset G$ is not normal! See exercise 3.2 from the manuscript. For more information about (possibly) surprising behaviour, one can take a look at `http://mathoverflow.net/questions/38891/is-there-a-connected-k-group-scheme-g-such-that-g-red-is-not-a-subgroup` and the following example by Laurent Moret-Bailly:

Over a field of characteristic $p > 0$, take for G the semidirect product $\alpha_p \rtimes \mathbb{G}\mathrm{m}$ where $\mathbb{G}\mathrm{m}$ acts on α_p by scaling. Then G is connected but $G_{\mathrm{red}} = \{0\} \times \mathbb{G}\mathrm{m}$ is not normal in G.

Example copied from: `http://mathoverflow.net/questions/161604/is-g-operatornamered-normal-in-g?rq=1`

5 Characteristic 0 group schemes are smooth

Let k be a field of characteristic 0. Let G/k be a group scheme that is locally of finite type.

Theorem 3 *G is reduced, hence G/k is smooth.*

Proof See Theorem 3.20 of the manuscript for a proof.

This result has some nice consequences.

- If G/k is finite, then it is étale.
- If G/k if finite, and k is algebraically closed, G/k is a constant group scheme.
- If G/k is finite, then it is a form of a constant group scheme.

6 Cartier duality for finite commutative group schemes

We only present Cartier duality over fields. For a more general picture, see the manuscript §3.21 and further.

Let k be a field. Let G/k be a finite commutative group scheme. To G we can attach the functor

$$\begin{aligned} G^D\colon\ \mathrm{Sch}_{/S}^{\mathrm{op}} &\longrightarrow \mathrm{Grp} \\ T &\longmapsto \mathrm{Hom}_{\mathrm{Grp}_{/S}}(G_T, \mathbb{G}\mathrm{m}_T) \end{aligned}$$

If G is commutative, finite, then G^D is representable.

To see this, first remark that since G is finite over k, G is affine. We can thus study G, by studying its Hopf algebra.

6.1 Hopf algebras

I am not going to discuss Hopf algebras in the generality that mathematical physicists would do.

The category of affine k-schemes is dual to the category of k-algebras. Hence a group object in the former corresponds to a cogroup object in the latter.

In particular, for an algebra A we get the following data

unit (algebra structure map)	$e\colon k \to A$
multiplication	$m\colon A \otimes_k A \to A$

and if A is a Hopf algebra, we moreover have

co-unit (augmentation map)	$\tilde{e}\colon A \to k$
co-multiplication	$\tilde{m}\colon A \to A \otimes_k A$
co-inverse	$\tilde{i}\colon A \to A$

I am not going to spell out what it means for A to be a co-commutative Hopf algebra, but you will just have to dualize all diagrams for group objects.

On k-algebras, use $(\_)^D$ as notation for the dualisation functor $\mathrm{Hom}(\_, k)$.

Lemma 2 *Let A be a co-commutative Hopf algebra over k. The dual data $(A^D, \tilde{e}^D, \tilde{m}^D, e^D, m^D, \tilde{i}^D)$ specifies a co-commutative k-Hopf algebra.*

Proof Draw all the diagrams for a co-commutative Hopf algebra. Reverse all the arrows. Remark that nothing happened, up to a permutation.

We return to the group scheme G/k. Recall that it is commutative and finite. Hence the global sections $\mathcal{O}_G(G)$ form a co-commutative Hopf algebra.

Theorem 4 *The Cartier dual G^D is represented by* $\mathrm{Spec}(A^D)$.

Proof Let R be any k-algebra. We have to show that $G^D(R)$ is naturally isomorphic to $\mathrm{Hom}_k(\mathrm{Spec}(R), \mathrm{Spec}(A^D))$.

Observe that

$$G^D(R) = \mathrm{Hom}_{\mathrm{GrpSch}_{/R}}(G_R, \mathbb{G}\mathrm{m}_R) \subset \mathrm{Hom}_R(R[x, x^{-1}], A \otimes_k R).$$

On the other hand,

$$\begin{aligned}\mathrm{Hom}_k(\mathrm{Spec}(R), \mathrm{Spec}(A^D)) &\cong \mathrm{Hom}_k(A^D, R)\\ &\cong \mathrm{Hom}_R(A^D \otimes_k R, R)\\ &\cong \mathrm{Hom}_R(A \otimes_k R^D, R).\end{aligned}$$

To make life easier, we now just write A for the R-Hopf algebra $A \otimes_k R$. So we want to prove that $\mathrm{Hom}_R(A^D, R)$ is canonically isomorphic to the subset of Hopf algebra homomorphisms of $\mathrm{Hom}_R(R[x, x^{-1}], A)$.

This latter subset is described as follows: A ring homomorphism f is determined by the image of x. It is a Hopf algebra homomorphism, precisely when $\tilde{m}(f(x)) = f(x) \otimes f(x)$.

So we get the set $\{a \in A^* \mid \tilde{m}(a) = a \otimes a\}$. From the diagrams for Hopf algebras, we see that if $a \in A$ satisfies $\tilde{m}(a) = a \otimes a$, then $\tilde{e}(a) \cdot a = a$, and $\tilde{i}(a) \cdot a = \tilde{e}(a)$. If $a \in A^*$, then $\tilde{e}(a) = 1$ by the first equation. If on the other hand $\tilde{e}(a) = 1$, then the second equation implies $a \in A^*$. Therefore

$$\{a \in A^* \mid \tilde{m}(a) = a \otimes a\} = \{a \in A \mid \tilde{m}(a) = a \otimes a \text{ and } \tilde{e}(a) = 1\}.$$

Reasoning from the other side, every R-module homomorphism $A^D \to R$ is an evaluation homomorphism

$$\begin{aligned} \mathrm{ev}_a \colon\ A^D &\longrightarrow R \\ \lambda &\longmapsto \lambda(a) \end{aligned}$$

If we want ev_a to be a ring homomorphism, it should satisfy

$$\begin{aligned} \mathrm{ev}_a(1) &= 1, & &\text{i.e. } \mathrm{ev}_a \circ \tilde{e}^D = \mathrm{id} \\ \mathrm{ev}_a(\lambda\mu) &= \mathrm{ev}_a(\lambda)\mathrm{ev}_a(\mu), & &\text{i.e. } \mathrm{ev}_a \circ \tilde{m}^D = \mathrm{ev}_a \cdot \mathrm{ev}_a. \end{aligned}$$

The first equation is equivalent with $\tilde{e}(a) = 1$. Indeed

$$\mathrm{ev}_a(\tilde{e}^D(\mathrm{id}_R)) = \tilde{e}(a).$$

The second equation demands

$$(\lambda \otimes \mu)(\tilde{m}(a)) = \lambda(a)\mu(a)$$

This is equivalent with $\tilde{m}(a) = a \otimes a$, which can be seen by letting λ and μ run through a dual basis of A.

Hence we are back at the set

$$\{a \in A \mid \tilde{m}(a) = a \otimes a \text{ and } \tilde{e}(a) = 1\}$$

which completes the proof.

7 Exercises

(1) Show that μ_n/k is unramified if $n \in k^*$.

(2) Show that μ_n/k is formally étale if $n \in k^*$.

(3) Let k be a field of characteristic p. Give a k-algebra A, such that $\alpha_p(A)$ is not trivial.

(4) Compute G^0 for $G = \mu_n/k$ (think about the characteristic of k).

(5) Compute the Cartier dual of $(\mathbb{Z}/n\mathbb{Z})_k$ for $n \in k^*$.