Erratum for my PhD thesis, entitled:

On ℓ-adic compatibility for abelian motives & the Mumford–Tate conjecture for products of K3 surfaces

September 8, 2017, *Johan Commelin*

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page 9, the last 3 lines before §0.4: "It is true that $G_\sigma(H^2(X_1) \oplus H^2(X_2))$ is a subgroup of $G_\sigma(H^2(X_1)) \times G_\sigma(H^2(X_2))$, but it may range from the graph of an isogeny to the the full product; and likewise on the ℓ-adic side."

This is not true. It is true if we replace $H^2(X_i)$ with $H^2(X_i)(1)$, with $i = 1, 2$. The reason that it is not true as written, is that $H^2(X_1)$ and $H^2(X_2)$ both have a factor $\mathbb{Q}(-1)$; and therefore $G_\sigma(H^2(X_1))$ and $G_\sigma(H^2(X_2))$ both have a quotient $\mathbb{G}_m$. Thus $G_\sigma(H^2(X_1)) \times G_\sigma(H^2(X_2))$ has a quotient $\mathbb{G}_m^2$, and $G_\sigma(H^2(X_1) \oplus H^2(X_2))$ projects diagonally into this torus of rank 2.

page 17, theorem 1.7: We need to assume that ρ_1 and ρ_2 are *semisimple* representations. (This assumption is satisfied in all instances where theorem 1.7 is applied.)

page 20, middle: "Let K be a field of characteristic 0. There are many fibre functors $\mathrm{Mot}_K \to \mathrm{Vect}_{\mathbb{Q}}$, but among those there is no natural choice presented to us."

We need to assume that K is embeddable into $\mathbb{C}$, otherwise it is not clear where these fibre functors come from. For the purposes of this thesis, it is sufficient to assume that K is finitely generated.

page 21, remark 2.8: "Similarly, if conjecture 2.7.2.a is true for all $M' \in \langle M \rangle^{\otimes}$, then conjecture 2.7.2.b is true for M."

This is true if $G_\ell^\circ(M)$ is a reductive group. We do not know in general whether this is true, although we do know that $G_\ell^\circ(M)$ is reductive if M is an abelian motive (see theorem 5.4). Alter the sentence to: "If conjecture 2.7.2.a is true for all $M' \in \langle M \rangle^{\otimes}$, and $G_\ell^\circ(M)$ is a reductive group, then conjecture 2.7.2.b is true for M."

page 23, line 4: "the group H_ℓ" should read "the group Γ_ℓ".

page 27, line 2 and 3 of §4.6: "Let V be a representation of G."

We need to assume that V is faithful.

page 30–31, theorem 5.5 and 5.6: There are several things wrong with these theorems and their proofs. Firstly, what the proofs attempt to prove is not the statement of the theorems but something that is slightly weaker. And secondly, the proofs contains some mistakes, so that they do not actually form a proof of what they attempt to prove. We fix this in the following errata.

page 30, statement of theorem 5.5: "Under Artin's comparison isomorphism (§2.4.2) we have $Z^\circ_\ell(A) \cong Z_\sigma(A) \otimes \mathbb{Q}_\ell$." Alter to: "Under Artin's comparison isomorphism (§2.4.2) we have an inclusion of centra $Z^\circ_\ell(A) \subset Z_\sigma(A) \otimes \mathbb{Q}_\ell$ that is an isomorphism on identity components $Z^\circ_\ell(A)^\circ = Z_\sigma(A)^\circ \otimes \mathbb{Q}_\ell$."

page 30, proof of theorem 5.5, point 1: We should assume that S is geometrically connected. We should assume that $\eta \in S$ is a Galois generic point. Without loss of generality we may replace K with a finitely generated extension of K; so we may and do assume that η and s are K-rational. The homomorphism $G_\sigma(\mathscr{A}_s)^{ab} \to G_\sigma(\mathscr{A}_\eta)^{ab}$ need not be injective: the use of "$\hookrightarrow$" is a mistake.

To make the rest of the argument in point 1 run through we need to argue that η is also a Hodge generic point, so that G is the generic Mumford–Tate group, etc...

Since η is a Galois generic point, we see that $G^\circ_\ell(\mathscr{A}_\eta)$ contains the geometric monodromy group. By theorem 5.2.2 we conclude that $G_\sigma(M)$ also contains the geometric monodromy group. In other words, η is also a Hodge generic point.

(Now that η is both a Galois generic point and a Hodge generic point, the existence of the diagonal arrows in the diagram at the bottom of page 30 is actually justified.)

page 31, statement of theorem 5.6: "Under Artin's comparison isomorphism (§2.4.2) we have $Z^\circ_\ell(M) \cong Z_\sigma(M) \otimes \mathbb{Q}_\ell$." Alter to: "Under Artin's comparison isomorphism (§2.4.2) we have an inclusion of centra $Z^\circ_\ell(M) \subset Z_\sigma(M) \otimes \mathbb{Q}_\ell$ that is an isomorphism on identity components $Z^\circ_\ell(M)^\circ = Z_\sigma(M)^\circ \otimes \mathbb{Q}_\ell$."

page 31, proof of theorem 5.6: In the diagram, all the vertical arrows should be inclusions. It is not known whether the first two vertical arrows are isomorphisms. The goal is to prove that the vertical arrow in the middle exists, and is an inclusion of algebraic groups of the same absolute rank. The vertical arrow on the left is an inclusion by theorem 5.5. This shows that the dotted vertical arrow (in the middle) exists, and is an inclusion. To see that $Z^\circ_\ell(M)$ and $Z_\sigma(M) \otimes \mathbb{Q}_\ell$ have the same absolute rank, take the left square of the diagram and pass to the identity components. Since the diagram commutes, and the left vertical arrow is now an isomorphism, we conclude that the inclusion $Z^\circ_\ell(M)^\circ \hookrightarrow Z_\sigma(M)^\circ$ is an isomorphism.

page 32, proof of lemma 5.8.4: "$Z_\ell(M) = Z_{\mathrm{mot},\ell}(M)$" should read "$Z_\ell(M) \subset Z_{\mathrm{mot},\ell}(M)$". In the diagram, the left vertical arrow should be an inclusion.

page 34, §6.4, line 1: "Let λ be a place of E." should read "Let λ be a finite place of E."

page 41, lemma 7.6: There is a natural ring homomorphism from the number field E to $\mathrm{End}(A) \otimes \mathbb{Q}$ and to $\mathrm{End}(T) \otimes \mathbb{Q}$. These homomorphisms are embeddings if A and T are non-trivial.

page 43, theorem 8.2: We need to assume that ρ_Λ and ρ'_Λ are quasi-compatible systems of *semisimple* Galois representations. (This assumption is satisfied in all instances where theorem 8.2 is applied.)

page 45, first 3 lines: It is only justified to call $T_x(\rho)$ a *Frobenius torus* if it is actually a torus. In general it is not known whether $F^n_{x,\rho}$ is a semisimple element. Thus the second and third line should be altered to read: "We denote this identity component with $T_x(\rho)$, and if there is an integer n such that $F^n_{x,\rho}$ acts semisimply, then we call $T_x(\rho)$ the *Frobenius torus* at x. (In this case the algebraic group $T_x(\rho)$ is indeed an algebraic torus, which means that $T_x(\rho)_{\bar{E}_\lambda} \cong \mathbb{G}^k_m$, for some $k \geq 0$.)"

Note: if M is an abelian motive, then $G^\circ_\ell(M)$ is reductive. It may be the case that not all of the $T_x(\rho)$ are algebraic tori, but corollary 8.9 guarantees that if $G_\ell(M)$ is connected, then $T_x(\rho)$ is a torus for a density 1 set of x.

page 46, §9.3: The reference to theorem 5.2.1 should be a reference to theorem 5.2.2.

page 46, §9.6, line 5: the sum should run over $p \geq \lceil n/2 \rceil$ instead of $p \geq \lfloor n/2 \rfloor$.

page 47, §9.7, line 5: the sum should run over $p \geq \lceil n/2 \rceil$ instead of $p \geq \lfloor n/2 \rfloor$.

page 47, proposition 9.8, line 5 of the proof: the sum should run over $p \geq \lceil n/2 \rceil$ instead of $p \geq \lfloor n/2 \rfloor$.

page 51, third-but-last line of §10.5: "$E_\ell = E \otimes \mathbb{Q}_\ell$ is a subfield of $\mathrm{End}(\mathcal{V}_{\ell,S})$"

The algebra $E_\ell = E \otimes \mathbb{Q}_\ell = \prod_{\lambda|\ell} E_\lambda$ is a product of local fields, but need not be a field. Hence "subfield" should read "subalgebra".

page 52, remark 10.9: Let M be an abelian motive over a finitely generated field K of characteristic 0. Let $E \subset \mathrm{End}(M)$ be a number field, and let Λ denote the set of finite places of E. Assume that $G_{\mathrm{mot},\sigma}(M)$ does not contain a factor of type D_k, for some $k \geq 4$. Then the results by Laskar in fact show not only that $H_{\mathcal{L}}(M)$ is a compatible system; but also that $H_\Lambda(M)$ is an E-rational compatible system.

page 54, remark 11.3, second line of the displayed formula: It is more natural to write a product of algebraic groups, instead of a direct sum. So "$\bigoplus$" should read "$\prod$".

page 59, §13.4, line 3: The domain of the spinorial norm should be $\mathrm{Cl}(\mathrm{V}, q)^\star$ instead of $\mathbb{G}_{\mathrm{m},\mathrm{Cl}(\mathrm{V},q)}$; just as in line 6.

page 67, one-but-last line: "$\mathrm{PGL}_{2,\mathbb{Q}}$" should read "$\mathrm{PGL}_{2,\mathbb{Q}_\ell}$".