The decomposition theorem

Notes by Johan Commelin Monday, the 10th of April, 2017

1. Introduction

The aim of this talk is to explain the main ideas in the proofs of two theorems: (*i*) the decomposition theorem over finite fields, see theorem 5.4.5 of [BBD]; and (*ii*) the hard Lefschetz theorem, see theorem 5.4.10 of [BBD]. We will now fix some notation, and then state these theorems.

1.1 Notation. We fix a finite field $\mathbb{F}_q$ of characteristic p, with q elements; and an algebraic closure $\mathbb{F}/\mathbb{F}_q$. Let ϕ_q denote the arithmetic Frobenius: $x \mapsto x^q$, and let Fr_q denote the geometric Frobenius: ϕ_q^{-1}. We adhere to the convention that objects over $\mathbb{F}_q$ will be decorated with a subscript '0', like so: $\diamondsuit_0$; and the corresponding object after extending scalars to $\mathbb{F}$ will be denoted by dropping the subscript, like so: $\diamondsuit$. In these notes we write $\mathrm{D}(X_0)$ for $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_0, \bar{\mathbb{Q}}_\ell)$, and likewise $\mathrm{D}(X)$ for $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X, \bar{\mathbb{Q}}_\ell)$.

1.2 Theorem. (Decomposition theorem; 5.4.5 of [BBD]) *Let $K_0 \in \mathrm{D}_m(X_0)$ be pure. Then*

$$K = \bigoplus_i {}^{\mathfrak{p}}\mathrm{H}^i(K)[-i].$$

To state the relative hard Lefschetz theorem, we need a short setup: Let $f\colon X_0 \to Y_0$ be a projective morphism. Let $\ell \in \mathrm{H}^2(X_0, \mathbb{Q}_\ell(1))$ be the first Chern class of a relatively ample invertible sheaf. For every object $K_0 \in \mathrm{D}(X_0)$, the class ℓ defines a morphism from K_0 to $K_0[2](1)$. Iteratively we obtain morphisms $\ell^n\colon K_0 \to K_0[2n](n)$. After applying f_* we get $\ell^n\colon f_*K_0 \to f_*K_0[2n](n)$, and $\ell^n\colon {}^{\mathfrak{p}}\mathrm{H}^i f_*K_0 \to {}^{\mathfrak{p}}\mathrm{H}^{i+2n} f_*K_0(n)$.

1.3 Theorem. (Relative hard Lefschetz theorem; 5.4.10 of [BBD]) *Let $f\colon X_0 \to Y_0$ be a projective morphism. Let $\ell \in \mathrm{H}^2(X_0, \mathbb{Q}_\ell(1))$ be the first Chern class of a relatively ample invertible sheaf. Let F_0 be a pure perverse sheaf on X_0. For every $i \geq 0$, the morphism*

$$\ell^i\colon {}^{\mathfrak{p}}\mathrm{H}^{-i} f_*F_0 \to {}^{\mathfrak{p}}\mathrm{H}^i f_*F_0\,(i)$$

is an isomorphism.

2. Recap from previous weeks

Recall the following results from Pol's talk.

2.1 Theorem. (5.1.2 of [BBD]) *The functor $F_0 \mapsto (F, \mathrm{F}_q^*)$ from perverse sheaves on X_0 to perverse sheaves on X endowed with an isomorphism $\mathrm{Fr}_q^* F \to F$ is fully faithful and the essential image is stable under extensions and subquotients.*

2.2 Theorem. (part of 5.1.14 of [BBD]) (*i*) *$f_!$, f^* respect $\mathrm{D}_{\leq w}$; (ii) $f^!$, f_* respect $\mathrm{D}_{\geq w}$.*

Recall the main result of Arne's talk: the weight filtration.

2.3 Theorem. (5.3.5 of [BBD]) *A mixed perverse sheaf F_0 on X_0 admits a unique increasing filtration W, the weight filtration, such that the graded pieces $\mathrm{Gr}_i^W F_0$ are pure of weight i. Every morphism $F_0 \to G_0$ is strictly compatible with these weight filtrations.*

2.4 Corollary. (5.3.7 of [BBD]) *For a mixed perverse sheaf F_0 on X_0 to be of weight $\leq w$, it is necessary and sufficient that that for every irreducible subvariety $Y_0 \subset X_0$ (of dimension $d = \dim(Y_0)$) there is a dense open $U_0 \subset Y_0$ such that $\mathrm{H}^{-d}(F_0)$ is pointwise of weight $\leq w - d$.*

3. First result

3.1 Fact. (5.1.15.(ii) and (iii) of [BBD]) *Let K_0 and L_0 be in $\mathrm{D}_m(X_0)$. Let w be an integer. Assume that $K_0 \in \mathrm{D}_{\leq w}(X_0)$, and $L_0 \in \mathrm{D}_{\geq w}(X_0)$. Then we have $\mathrm{Hom}^i(K, L)^F = 0$ for $i > 0$. In particular, for $i > 0$, the morphism $\mathrm{Hom}^i(K_0, L_0) \to \mathrm{Hom}^i(K, L)$ is the zero map. If $L_0 \in \mathrm{D}_{>w}(X_0)$, then $\mathrm{Hom}^i(K_0, L_0) = 0$.*

3.2 Theorem. (5.3.8 of [BBD]) *Let F_0/X_0 be a pure perverse sheaf. Then the perverse sheaf F/X is the direct sum of simple perverse sheaves (which must then be of the form $j_{!*}L[d]$, where $j\colon U \to X$ is the inclusion of a smooth connected open that is pure of dimension d and L is an irreducible lisse $\bar{\mathbb{Q}}_\ell$-sheaf on U).*

Proof. Let F' be the direct sum of the simple perverse subsheaves of F; it is the largest semisimple subobject of F, and it suffices to show that $F' = F$. By this maximality property, we know that F' is stable under Frobenius and thus comes from a subsheaf $F_0' \subset F$. Now consider the extension $0 \to F_0' \to F_0 \to F_0'' \to 0$ and use fact 3.1 to conclude that $F \cong F' \oplus F''$. By the maximality of F' we conclude that $F'' = 0$. Thus $F' = F$. QED

3.3 Caveat. (Remark 5.3.10 of [BBD]) So far, everything we have done was with $\bar{\mathbb{Q}}_\ell$-coefficients, but it could also have been done with $\mathbb{Q}_\ell$-coefficients (or coefficients in some other ℓ-adic field); although the linear algebra involved is a lot harder. However, the following result is not true with $\mathbb{Q}_\ell$-coefficients; we really need $\bar{\mathbb{Q}}_\ell$.

3.4 Theorem. (5.3.11 of [BBD]) *Let F_0 be a pure perverse sheaf on X_0, let $j\colon U_0 \to X_0$ be an open, and let $i\colon Z_0 \to X_0$ be the complementary closed. Then the perverse sheaf F_0 admits a unique decomposition*

$$F_0 = j_{!*}F_0' \oplus i_*F_0''$$

as follows: (i) the adjunction morphism ${}^{\mathrm{p}}j_!j^*F_0 \to F_0$ factors via the quotient $j_{!*}j^*F_0$ of ${}^{\mathrm{p}}j_!j^*F_0$, while ${}^{\mathrm{p}}j_*j^*F_0 \to F_0$ factors via the subobject $j_{!*}j^*F_0$ of ${}^{\mathrm{p}}j_*j^*F_0$ (so we should take $F_0' = j^*F_0$); and (ii) the composition of the adjunction morphisms

$$i_*{}^{\mathrm{p}}i^!F_0 \to F_0 \to i_*{}^{\mathrm{p}}i^*F_0$$

is an isomorphism (so we should take $F_0'' = i^*F_0$).

Proof. The uniqueness follows from the fact that $\mathrm{Hom}(j_{!*}F_0', i_*F_0'') = 0 = \mathrm{Hom}(i_*F_0'', j_{!*}F_0')$. It suffices to prove the existence over X (by theorem 2.1). By theorem 3.2 we may assume that F is simple. Now the result is evident. QED

4. The decomposition theorem

The folloing result is the perverse analogue of corollary 2.4.

4.1 Theorem. (5.4.1 of [BBD]) *For $K_0 \in \mathrm{D}_m(X_0)$ to be of weight $\leq w$ (resp. $\geq w$), it is necessary and sufficient that all the perverse sheaves ${}^{\mathfrak{p}}\mathrm{H}^i K_0$ are of weight $\leq w+i$ (resp. $\geq w+i$).*

Proof. The respective assertion is dual to the non-respective assertion, to which we limit our attention.

If the triangle $A_0 \longrightarrow B_0 \longrightarrow C_0 \xrightarrow{+1} \cdots$ is distinguished, and if A_0 and C_0 are of weight $\leq w$, then so is B_0. Thus the sufficient condition follows: use the triangle ${}^{\mathfrak{p}}\tau_{<i}K_0 \longrightarrow {}^{\mathfrak{p}}\tau_{\leq i}K_0 \longrightarrow ({}^{\mathfrak{p}}\mathrm{H}^i K_0)[-i] \xrightarrow{+1} \cdots$ and apply induction to i.

Suppose that K_0 is of weight $\leq w$. We prove by induction to i that the sheaf ${}^{\mathfrak{p}}\mathrm{H}^i K_0$ is of weight $\leq w+i$. For i sufficiently large, the ${}^{\mathfrak{p}}\mathrm{H}^i K_0$ are zero, and the assertion is trivial. Suppose then that the ${}^{\mathfrak{p}}\mathrm{H}^i K_0$ are of weight $\leq w+i$ for $i > n$, so that the truncation ${}^{\mathfrak{p}}\tau_{>n}K_0$ is of weight $\leq w$ (by the sufficient condition proven above). We prove that ${}^{\mathfrak{p}}\mathrm{H}^n K_0$ is of weight $\leq w+n$, and our main weapon is corollary 2.4.

For simplicity we assume that $w = n = 0$; one may reduce to this case by twisting and shifting.

The distinguished triangle ${}^{\mathfrak{p}}\tau_{>0}K_0[-1] \longrightarrow {}^{\mathfrak{p}}\tau_{\leq 0}K_0 \longrightarrow K_0 \xrightarrow{+1} \cdots$ shows that ${}^{\mathfrak{p}}\tau_{\leq 0}K_0$ is of weight ≤ 0 (once again, by the sufficient condition proven above). The triangle ${}^{\mathfrak{p}}\tau_{<0}K_0 \longrightarrow {}^{\mathfrak{p}}\tau_{\leq 0}K_0 \longrightarrow {}^{\mathfrak{p}}\mathrm{H}^0 K_0 \xrightarrow{+1} \cdots$ gives the exact sequence

$$\mathrm{H}^{-d}({}^{\mathfrak{p}}\tau_{\leq 0}K_0) \longrightarrow \mathrm{H}^{-d}({}^{\mathfrak{p}}\mathrm{H}^0 K_0) \longrightarrow \mathrm{H}^{-d+1}({}^{\mathfrak{p}}\tau_{<0}K_0).$$

If Y_0 is an irreducible subvariety of dimension d, then there exists a smooth dense open $U_0 \subset Y_0$ such that the restriction of $\mathrm{H}^{-d+1}({}^{\mathfrak{p}}\tau_{<0}K_0)$ to U_0 is zero. Hence the pointwise weights of $\mathrm{H}^{-d}({}^{\mathfrak{p}}\mathrm{H}^0 K_0)$ are $\leq -d$ on U_0. By corollary 2.4, ${}^{\mathfrak{p}}\mathrm{H}^0 K_0$ is of weight ≤ 0, which proves the result. QED

4.2 Corollary. *For $K_0 \in \mathrm{D}_m(X_0)$ to be pure of weight w, it is necessary and sufficient that all the perverse sheaves ${}^{\mathfrak{p}}\mathrm{H}^i K_0$ are pure of weight $w+i$.*

Proof of the decomposition theorem (1.2). By corollary 4.2 and fact 3.1 the morphism of degree 1 in the distinguished triangle ${}^{\mathfrak{p}}\tau_{<i}K_0 \longrightarrow {}^{\mathfrak{p}}\tau_{\leq i}K_0 \longrightarrow ({}^{\mathfrak{p}}\mathrm{H}^i K_0)[-i] \xrightarrow{+1} \cdots$ has trivial image in $\mathrm{Hom}^1({}^{\mathfrak{p}}\mathrm{H}^i K[-i], {}^{\mathfrak{p}}\tau_{<i}K)$ (because both $({}^{\mathfrak{p}}\mathrm{H}^i K_0)[-i]$ and ${}^{\mathfrak{p}}\tau_{<i}K_0$ have weight w). Hence we get short exact sequences

$$0 \longrightarrow {}^{\mathfrak{p}}\tau_{<i}K \longrightarrow {}^{\mathfrak{p}}\tau_{\leq i}K \longrightarrow ({}^{\mathfrak{p}}\mathrm{H}^i K)[-i] \longrightarrow 0$$

that must be trivial extensions. In other words, we get a decompositions ${}^{\mathfrak{p}}\tau_{\leq i}K \cong {}^{\mathfrak{p}}\tau_{<i}K \oplus ({}^{\mathfrak{p}}\mathrm{H}^i K)[-i]$, which completes the proof (by induction). QED

5. The hard Lefschetz theorem

Let us recall the statement: Let $f\colon X_0 \to Y_0$ be a projective morphism. Let $\ell \in \mathrm{H}^2(X_0, \mathbb{Q}_\ell(1))$ be the first Chern class of a relatively ample invertible sheaf. For every object $K_0 \in \mathrm{D}(X_0)$, the class ℓ defines a morphism from K_0 to $K_0[2](1)$. Iteratively we obtain morphisms $\ell^n\colon K_0 \to K_0[2n](n)$. After applying f_* we get $\ell^n\colon f_*K_0 \to f_*K_0[2n](n)$, and $\ell^n\colon {}^{\mathfrak{p}}\mathrm{H}^i f_*K_0 \to {}^{\mathfrak{p}}\mathrm{H}^{i+2n} f_*K_0(n)$.

5.1 Theorem. (Relative hard Lefschetz theorem; 5.4.10 of [BBD]) *Let $f\colon X_0 \to Y_0$ be a projective morphism. Let $\ell \in \mathrm{H}^2(X_0, \mathbb{Q}_\ell(1))$ be the first Chern class of a relatively ample invertible sheaf. Let F_0 be a pure perverse sheaf on X_0. For every $i \geq 0$, the morphism*

$$\ell^i\colon {}^{\mathfrak{p}}\mathrm{H}^{-i} f_* F_0 \to {}^{\mathfrak{p}}\mathrm{H}^{i} f_* F_0\,(i)$$

is an isomorphism.

Eventually, the proof will go by induction on i; the case $i = 0$ is trivial. But first we need a lot of preparations.

Note that the statement is local on Y_0. Thus we may shrink Y_0, and upon replacing ℓ be some multiple we may and do assume (i) that f factors as $X_0 \hookrightarrow \mathbb{P}^d_0 \times Y_0 \to Y_0$, for some suitable $d \geq 0$; and (ii) that ℓ is the first Chern class of $\mathcal{O}(1)$.

Let $\check{\mathbb{P}}^d_0$ be the dual projective space of $\mathbb{P}^d_0$, and put $Y'_0 = \check{\mathbb{P}}^d_0 \times Y_0$, and $X'_0 = \check{\mathbb{P}}^d_0 \times X_0$. Let $H_0 \subset X'_0$ be the universal family of hyperplane sections parameterised by Y'_0: the fibre of H/Y' above $(a, y) \in Y'$ is the hyperplane section $X_y \cap a$ of the fibre X_y.

We denote the projections $X'_0 \to X_0$ and $Y'_0 \to Y_0$ by u and the map $X'_0 \to Y'_0$ that is induced by f is again denoted by f. This might cause some confusion, but in what follows it will be clear from the context which map is meant. In particular, note that $fu = uf$. We write h for the projection $H_0 \to Y'_0$, and v for the inclusion $H_0 \to X'_0$.

$$\begin{array}{ccccc} X_0 & \xleftarrow{\;u\;} & X'_0 & \xleftarrow{\;v\;} & H_0 \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle f}\big\downarrow & \swarrow{\scriptstyle h} & \\ Y_0 & \xleftarrow{\;u\;} & Y'_0 & & \end{array}$$

Key facts for proving hard Lefschetz: (i) the functors $u^*[d]$ are fully faithful, so we can work on X' and Y'; and (ii) the largest perverse subsheaf of $F \in \mathrm{D}(Y')$ that comes from Y is $u^*({}^{\mathfrak{p}}\mathrm{H}^{-d} u_* F)[d]$.

5.2 Lemma. (5.4.11 of [BBD]) *Let K be in ${}^{\mathfrak{p}}\mathrm{D}^{\geq 0}(X)$.*

(i) Then we have $u^* f_* K \xrightarrow{\sim} f_* u^* K$, and therefore $(u^* {}^{\mathfrak{p}}\mathrm{H}^i f_* K)[d] \xrightarrow{\sim} {}^{\mathfrak{p}}\mathrm{H}^{i+d} f_* u^* K$.

(ii) For $i < d - 1$, we have ${}^{\mathfrak{p}}\mathrm{H}^i f_* u^* K \xrightarrow{\sim} {}^{\mathfrak{p}}\mathrm{H}^i h_* (uv)^* K$.

(iii) We have ${}^{\mathfrak{p}}\mathrm{H}^{d-1} f_* u^* K \hookrightarrow {}^{\mathfrak{p}}\mathrm{H}^{d-1} h_* (uv)^* K$, and ${}^{\mathfrak{p}}\mathrm{H}^{d-1} f_* u^* K$ is identified with the largest perverse subsheaf of ${}^{\mathfrak{p}}\mathrm{H}^{d-1} h_* (uv)^* K$ that comes from Y.

Proof. The morphisms u are smooth of relative dimension d. Hence the functors $u^*[d]$ are t-exact (by 4.2.4 of [BBD]), which proves (i). Write U for the complement of H in X', and denote the projection $U \to Y'$ by g. It is an affine morphism, and $u^* K$ is in ${}^{\mathfrak{p}}\mathrm{D}^{\geq d}(X')$, hence $g_!(u^* K|_U)$ is in ${}^{\mathfrak{p}}\mathrm{D}^{\geq d}(Y')$ (by 4.1.2 of [BBD]). Now consider the triangle

$$g_!(u^* K|_U) \longrightarrow f_* u^* K \longrightarrow h_* (uv)^* K \xrightarrow{+1} \cdots$$

and remark that ${}^{\mathfrak{p}}\mathrm{H}^i(g_!(u^* K|_U)) = 0$ for $i < d$. This proves (ii) and the first assertion (injectivity) of (iii).

The second assertion of (iii) is crucial for what follows; but its proof is rather technical. We skip this part of the proof. QED

Proof of theorem 5.1. We have made all necessary preparations; now is the time to start the actual proof.

Step 1. Consider the restriction morphism

$$(u^* {}^{\mathfrak{p}}\mathrm{H}^{-i} f_* F)[d] = {}^{\mathfrak{p}}\mathrm{H}^{-i} f_* (u^* F[d]) \to {}^{\mathfrak{p}}\mathrm{H}^{-i+1} h_* ((uv)^* F[d-1]).$$

Now we need non-trivial input from étale cohomology, the so-called Gysin morphism: as a little miracle, there is a morphism

$$^{\mathfrak{p}}\mathrm{H}^j h_*((uv)^*F[d-1]) \to u^*(^{\mathfrak{p}}\mathrm{H}^{j+1} f_*F)[d](1).$$

Here we use that $v\colon H \hookrightarrow X'$ has codimension 1. Now we compose these to morphisms (with $j = -i+1$) and obtain a morphism

$$(u^{*\,\mathfrak{p}}\mathrm{H}^{-i} f_*F)[d] \to u^*(^{\mathfrak{p}}\mathrm{H}^{-i+2} f_*F)[d](1).$$

It turns out that this morphism is precisely the $u^*[d]$ of the morphism ℓ that occurs in the statement of theorem 5.1.

Let us focus on the case $i = 1$. By lemma 5.2 we know that $(u^{*\,\mathfrak{p}}\mathrm{H}^{-1} f_*F)[d]$ is the largest subobject of $^{\mathfrak{p}}\mathrm{H}^0 h_*((uv)^*F[d-1])$ that comes from Y. Dually we know that $u^*(^{\mathfrak{p}}\mathrm{H}^1 f_*F)[d](1)$ is the largest quotient of $^{\mathfrak{p}}\mathrm{H}^0 h_*((uv)^*F[d-1])$ that comes from Y.

Since F_0 is pure, so is $(uv)^*F_0$ (since $u^! = u^*[2d](d)$, and $v_! = v_*$; now use theorem 2.2), and hence $h_*((uv)^*F_0)$ (h is proper). By corollary 4.2 we conclude that $^{\mathfrak{p}}\mathrm{H}^0 h_*((uv)^*F_0)$ is pure, and therefore $^{\mathfrak{p}}\mathrm{H}^0 h_*((uv)^*F[d-1])$ is semi-simple (theorem 3.2), which renders its largest subobject that comes from Y isomorphic to its largest quotient that comes from Y. Therefore $u^*[d]$ applied to ℓ is an isomorphism; but then ℓ is an isomorphism itself.

Step 2. We proceed by induction on i. For $i \geq 1$, consider the $u^*[d]$ of ℓ^{i+1} as the composition

$$(u^{*\,\mathfrak{p}}\mathrm{H}^{-i-1} f_*F)[d] \longrightarrow (u^{*\,\mathfrak{p}}\mathrm{H}^{-i} f_*F)[d-1] \xrightarrow{\ell^i} (u^{*\,\mathfrak{p}}\mathrm{H}^{i} f_*F)[d-1](i) \longrightarrow (u^{*\,\mathfrak{p}}\mathrm{H}^{i+1} f_*F)[d](i+1),$$

where the last morphism is again the Gysin morphism.

By lemma 5.2 the first morphism is an isomorphism, and the final morphism (the Gysin map) is an isomorphism for dual reasons. Now apply the induction hypothesis to the sheaf $(uv)^*F_0[d-1]$ and the morphism $h\colon H_0 \to Y_0'$ to deduce that the morphism in the middle is also an isomorphism. This proves the result. QED