A theorem of Bondal and Orlov

Notes by Johan Commelin Friday, the 13th of October, 2017

-1. Introduction

Let X be a smooth projective variety of dimension d over K. In the previous talk we have seen how to attach a triangulated category $\mathrm{D^b}(X)$ to X: namely, the bounded derived category of coherent sheaves on X.

In this talk we will focus on a theorem of Bondal and Orlov, that shows that $\mathrm{D^b}(X)$ is a complete invariant of X as soon as the canonical bundle $\omega_X = \bigwedge^d \Omega^1_X$ (or its dual $\omega_X^\vee$) is ample. On, the other hand, if neither ω_X nor $\omega_X^\vee$ is ample, then there may be another smooth projective variety Y over K such that $\mathrm{D^b}(X) \cong \mathrm{D^b}(Y)$. We may then use such an equivalence as a sort of 'bridge'.

For purposes of exposition, we assume that the ground field K is algebraically closed. This simplifies some of the proofs, although all statements are valid over arbitrary fields.

0. Our todo list for today

1. canonical models
2. Serre functors
3. dimension lemma
4. simple objects and point-like objects
5. invertible objects
6. the theorem of Bondal and Orlov
7. (ample sequences)?
8. (a description of $\mathrm{Aut}(\mathrm{D^b}(X))$ when ω_X or $\omega_X^\vee$ is ample)?

1. The canonical model

Let X be a smooth projective variety over K. Let ω_X be the canonical bundle $\bigwedge^d \Omega^1_X$, where $d = \dim(X)$. Form the graded algebra $B_X = \bigoplus_{k\in\mathbb{Z}} H^0(X, \omega_X^k)$, where multiplication is induced from the algebra structure on the exterior algebra $\bigoplus_{k\in\mathbb{Z}} \omega_X^k$. Denote with B_X^+ the subalgebra $\bigoplus_{k\in\mathbb{Z}_{\geq 0}} H^0(X, \omega_X^k)$. The algebra B_X^+ is called the *canonical ring*, and $\mathrm{Proj}(B_X^+)$ is called the *canonical model* of X: there is a natural morphism $X \to \mathrm{Proj}(B_X^+)$.

1.1 Remark. (*i*) The variety $\mathrm{Proj}(B_X^+)$ only depends on the birational equivalence class of X. For example, a blowup of X will have the same canonical model as X. (*ii*) The dimension $\mathrm{kod}(X)$ of $\mathrm{Proj}(B_X^+)$ is called the *Kodaira dimension* of X; we have $\mathrm{kod}(X) \leq \dim(X)$, and $\mathrm{kod}(X) := -\infty$ if $\mathrm{Proj}(B_X^+)$ is empty. (*iii*) Usually the variety $\mathrm{Proj}(B_X^+)$ has singularities. (*iv*) The map $X \to \mathrm{Proj}(B_X^+)$ is an isomorphism if and only if ω_X is ample. (*v*) Together, the previous remarks show that it is a strong condition to require that ω_X is ample.

In this talk we require usually require that ω_X or $\omega_X^\vee$ is ample. This is still a strong condition. We therefore do not look at the map $X \to \mathrm{Proj}(B_X^+)$ but at the natural map $X \to \mathrm{Proj}(B_X)$. This map is an isomorphism if and only if ω_X or $\omega_X^\vee$ is ample.

1.2 Lemma. *Let X be a smooth projective variety over K. Then B_X may be recovered from the data $(\mathrm{D^b}(X), \mathcal{O}_X, \omega_X)$, where we consider $\mathrm{D^b}(X)$ as K-linear category. In particular, if ω_X or $\omega_X^\vee$ is ample, we recover X.*

Proof. Observe that $H^0(X, \omega_X^k) = \mathrm{Hom}(O_X, \omega_X^k)$. Thus $B_X = \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}(O_X, \omega_X^k)$ may be recovered from $(\mathrm{D^b}(X), \mathcal{O}_X, \omega_X)$. QED

2. Serre functors

2.1 Definition (def 1.28 of [H]). Let $\mathcal{A}$ be a K-linear category. A *Serre functor* is a K-linear equivalence $S\colon \mathcal{A} \to \mathcal{A}$ such that for any two objects $A, B \in \mathcal{A}$ there exists an isomorphism

$$\eta_{A,B}\colon \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(B, S(A))^\vee$$

(of K-vector spaces) that is functorial in A and B.

2.2 Example. Let us return to the smooth projective variety X of dimension d over K. Serre duality provides us with an example of a Serre functor on $\mathrm{D^b}(X)$: $S_X(\mathcal{F}^\bullet) = \mathcal{F}^\bullet \otimes \omega_X[d]$. For this reason, the canonical bundle ω_X is also called the *dualizing sheaf.*

2.3 Lemma. *Let $\mathcal{A}$ be a K-linear category. Let S_1 and S_2 be Serre functors on $\mathcal{A}$. Then $S_1 \cong S_2$.*

Proof. This follows immediately from the definition and the Yoneda lemma. QED

2.4 Lemma (lem 1.30 of [H]). *Let $\mathcal{A}$ and $\mathcal{B}$ be two K-linear categories. Let $S_\mathcal{A}$ (resp. $S_\mathcal{B}$) be a Serre functor on $\mathcal{A}$ (resp. $\mathcal{B}$). If $F\colon \mathcal{A} \to \mathcal{B}$ is a K-linear equivalence, then F commutes with the Serre functors $S_\mathcal{A}$ and $S_\mathcal{B}$. In other words: $F \circ S_\mathcal{A} \cong S_\mathcal{B} \circ F$.*

Proof. This follows from lemma 2.3 and transport of structure. QED

2.5 Lemma. *Let X be a smooth projective variety over K. Then B_X may be recovered from the data $(\mathrm{D^b}(X), \mathcal{O}_X)$, where we consider $\mathrm{D^b}(X)$ as graded K-linear category. In particular, if ω_X or $\omega_X^\vee$ is ample, we recover X.*

Proof. It follows from lemma 2.3 that up to isomorphism there is a unique Serre functor S on $\mathrm{D^b}(X)$. Now observe that $\omega_X \cong S(\mathcal{O})[-\dim X]$. (Here we use the graded structure on $\mathrm{D^b}(X)$.) We conclude by lemma 1.2. QED

3. The dimension lemma

With this bit of knowledge on Serre functors under our belt, we can also prove that one can recover the dimension of X from just $\mathrm{D^b}(X)$.

3.1 LEMMA (prp 4.1 of [H]). *Let X and Y be smooth projective varieties over K. If there exists a derived equivalence*

$$\mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(Y)$$

then $\dim(X) = \dim(Y)$ *and the canonical bundles ω_X and ω_Y are of the same order.*

Proof. Let $x \in X$ be a closed point. With $K(x)$ we denote the skyscraper sheaf at x. (It is the sheaf F such that $F(U) = k$ if $x \in U$, and $F(U) = 0$ otherwise. In other words $i_* k$, where $i\colon \{x\} \hookrightarrow X$ is the inclusion.) Then

$$K(x) \cong K(x) \otimes \omega_X \cong S_X(K(x))[-\dim(X)].$$

Hence we get

$$F(K(x)) \cong F(S_X(K(x))[-\dim(X)]) \cong F(S_X(K(x)))[-\dim(X)]$$

since F is exact, and by lemma 2.4 we may continue with

$$F(S_X(K(x)))[-\dim(X)] \cong S_Y(F(K(x)))[-\dim(X)] = F(K(x)) \otimes \omega_Y[\dim(Y) - \dim(X)].$$

Now observe that $F(K(x))$ is a non-trivial(!) bounded complex, since F is an equivalence. Therefore

$$\mathcal{H}^i(F(K(x))) \cong \mathcal{H}^i(F(K(x)) \otimes \omega_Y[\dim(Y) - \dim(X)]) \cong \mathcal{H}^{i+\dim(Y)-\dim(X)}(F(K(x))) \otimes \omega_Y$$

shows that $\dim(Y) - \dim(X) = 0$, for otherwise the complex would not be bounded. In other words, X and Y have the same dimension, say d.

Suppose that $\omega_X^k \cong \mathcal{O}_X$. The $S_X^k[-kd] = \mathrm{id}$. Hence $S_Y^k[-kd] \cong F \circ S_X^k[-kd] \circ F^{-1} = \mathrm{id}$. This means that $\omega_Y^k \cong \mathcal{O}_Y$. We conclude that ω_X and ω_Y are of the same order. QED

4. Simple objects and point-like objects

4.1 DEFINITION (def 4.3 of [H]). Let $\mathcal{D}$ be a K-linear triangulated category with a Serre functor S And object $P \in \mathcal{D}$ is called *point-like of codimension d* if

1. $S(P) \cong P[d]$;
2. $\mathrm{Hom}(P, P[i]) = 0$, for $i < 0$; and
3. P is *simple*, that is: $\mathrm{Hom}(A, A)$ is a field.

4.2 EXAMPLE. Let X be a smooth projective variety of dimension d over K. Let $x \in X$ be a closed point. Then the skyscraper sheaf $K(x) \in \mathrm{D}^{\mathrm{b}}(X)$ is point-like of codimension d.

Assume that $\omega_X \cong \mathcal{O}_X$. (This condition is satisfied, for example, if X is an abelian variety or a K3 surface.) Let F be a simple sheaf on X. Then F defines a point-like object in $\mathrm{D}^{\mathrm{b}}(X)$.

4.3 LEMMA (lem 4.5 of [H]). *Let X be a smooth projective variety over K. Suppose that $\mathcal{F}^\bullet$ is a simple object of $\mathrm{D}^{\mathrm{b}}(X)$ with zero-dimensional support. If $\mathrm{Hom}(\mathcal{F}^\bullet, \mathcal{F}^\bullet[i]) = 0$, for all $i < 0$, then $\mathcal{F}^\bullet \cong K(x)[m]$ for some closed point $x \in X$ and some integer m.*

Proof. If the support of $\mathcal{F}^\bullet$ is not irreducible, then $\mathcal{F}^\bullet$ is not simple. (This is clear for sheaves, and by an induction argument using distinguished triangles, one may also deduce this for bounded complexes.) Hence there is a closed point $x \in X$ such that $\mathcal{F}^\bullet$ is supported on $\{x\}$.

Let i^+ be the maximal integer i such that $\mathcal{H}^i(\mathcal{F}^\bullet) \neq 0$, and i^- be the minimal integer i such that $\mathcal{H}^i(\mathcal{F}^\bullet) \neq 0$. Note that $\mathcal{H}^i(\mathcal{F}^\bullet)$ is a finite module over the Noetherian local ring $\mathcal{O}_x$, and thus there is a non-trivial homomorphism $\mathcal{H}^{i^+}(\mathcal{F}^\bullet) \to \mathcal{H}^{i^-}(\mathcal{F}^\bullet)$. This gives a non-trivial composition

$$\mathcal{F}^\bullet[i^+] \to \mathcal{H}^{i^+}(\mathcal{F}^\bullet) \to \mathcal{H}^{i^-}(\mathcal{F}^\bullet) \to \mathcal{F}^\bullet[i^-].$$

By assumption, this implies $i^- - i^+ \geq 0$, and hence $i^- = i^+$. Write i for $i^- = i^+$. We have now showed that $\mathcal{F}^\bullet[i]$ is isomorphic to a coherent sheaf F with support on x. There is a natural quotient map $F \to K(x)$, and this must be an isomorphism, since F is simple. QED

4.4 Proposition ([BO]; prp 4.6 of [H]). *Let X be a smooth projective variety over K. Assume that ω_X or $\omega_X^\vee$ is ample. Then every point-like object of $\mathrm{D^b}(X)$ is isomorphic to $K(x)[m]$ for some closed point $x \in X$ and some integer m.*

Proof. Write d for the dimension of X. Let P be a point-like object of $\mathrm{D^b}(X)$ of codimension c. Abbreviate $\mathcal{H}^i(P)$ by $\mathcal{H}^i$. By the first condition on P we find $\mathcal{H}^i \otimes \omega_X[d] \cong \mathcal{H}^i[c]$, which implies $c = d$ and $\mathcal{H}^i \otimes \omega \cong \mathcal{H}^i$.

For any coherent sheaf F, the Hilbert polynomial $P_F(k) = \chi(F \otimes \omega_X^k)$ has degree $\dim \operatorname{supp}(F)$. We have just computed that $P_{\mathcal{H}^i}(k)$ is constant, and therefore $\mathcal{H}^i$ has zero-dimensional support. Now apply lemma 4.3. QED

4.5 Lemma (prp 3.17 of [H]). *Let X be a smooth projective variety over K. Then the objects $K(X)$, with $x \in X$ a closed point, form a spanning class for $\mathrm{D^b}(X)$.*

Proof. This means that for every object $\mathcal{F}^\bullet$ in $\mathrm{D^b}(X)$ there exist closed points $x_1, x_2 \in X$, and integers $i_1, i_2 \in \mathbb{Z}$ such that

$$\mathrm{Hom}(\mathcal{F}^\bullet, K(x_1)[i_1]) \neq 0 \neq \mathrm{Hom}(K(x_2)[i_2]\mathcal{F}^\bullet).$$

By Serre duality we only have to find x_1 and i_1. Let m be the maximal integer i such that $\mathcal{H}^i = \mathcal{H}^i(\mathcal{F}^\bullet) \neq 0$. Let x_1 be a closed point in $\operatorname{supp}(\mathcal{H}^m)$. Then there is a non-trivial map $\mathcal{H}^m \to K(x_1)$, and we have $\mathrm{Hom}(\mathcal{F}^\bullet, K(x_1)[-m]) = \mathrm{Hom}(\mathcal{H}^m, K(x_1)) \neq 0$. Thus we win, by taking $i_1 = -m$. QED

5. Invertible objects

5.1 Definition (def 4.8 of [H]). Let $\mathcal{D}$ be a K-linear triangulated category, with a Serre functor S. An object $L \in \mathcal{D}$ is called *invertible* if for every point-like object $P \in \mathcal{D}$ there exists an integer n such that

$$\mathrm{Hom}(L, P[i]) = \begin{cases} K & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

5.2 Proposition ([BO]; prp 4.9 of [H]). *Let X be a smooth projective variety over K. Every invertible object in $\mathrm{D^b}(X)$ is of the form $L[m]$, with L a line bundle on X, and $m \in \mathbb{Z}$. Conversely, if ω_X or $\omega_X^\vee$ is ample, then for every line bundle L on X, and every $m \in \mathbb{Z}$, the object $L[m] \in \mathrm{D^b}(X)$ is invertible.*

Proof. Let L be an invertible object in $\mathrm{D^b}(X)$. Let m be the maximal integer i such that $\mathcal{H}^i = \mathcal{H}^i(L) \neq 0$. There is a natural map $L \to \mathcal{H}^m[-m]$. Let x be a closed point in $\mathrm{supp}(\mathcal{H}^m)$. There exists a non-trivial map $\mathcal{H}^m \to K(x)$. Since $K(x)[-m]$ is point-like, and we know that

$$\mathrm{Hom}(L, K(x)[-m]) = \mathrm{Hom}(\mathcal{H}^m, K(x)) \neq 0$$

we conclude that $\mathrm{Hom}(L, K(x)[i]) = 0$ for all $i \neq -m$.

The strategy of the rest of the proof is as follows:

1. Use spectral sequences and commutative algebra to show that $\mathcal{H}^m$ is locally free at x.
2. Deduce that $\mathrm{supp}(\mathcal{H}^m)$ is an irreducible subset of X, hence all of X.
3. This means that $\mathcal{H}^m$ is a line bundle.
4. The rank of $\mathcal{H}^m$ must be 1, by the defining property of invertible objects, hence $\mathcal{H}^m$ is a line bundle.
5. Use more spectral sequence arguments to deduce that $\mathcal{H}^i$ vanishes for $i < m$.

For the second part of the proposition, let L be a line bundle on X, and $m \in \mathbb{Z}$. Let $P \in \mathrm{D^b}(X)$ be a point-like object. By proposition 4.4 we know that $P = K(x)[n]$, for some closed point $x \in X$ and some $n \in \mathbb{Z}$. We compute

$$\mathrm{Hom}(L[m], P[i]) = \mathrm{Hom}(\mathcal{O}_X, L^\vee(x)[i+n-m]) = H^{i+n-m}(X, L^\vee(x)).$$

Now, the latter cohomology group is trivial, unless $i = m - n$, and then it is $H^0(X, L^\vee(x)) = K$. QED

6. A theorem of Bondal and Orlov

6.1 Theorem ([BO]; thm 4.11 of [H]). *Let X and Y be smooth projective varieties over K. Assume that ω_X or $\omega_X^\vee$ is ample. If there exists a graded equivalence $\mathrm{D^b}(X) \cong \mathrm{D^b}(Y)$, then $X \cong Y$. In particular, ω_Y or $\omega_Y^\vee$ is also ample.*

Proof. Denote the equivalence $\mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ with F. The graded equivalence preserves point-like objects and invertible objects. By we know that $\mathcal{O}_X$ is invertible, and therefore $F(\mathcal{O}_X)$ is of the form $L[m]$ for some line bundle L on Y, and $m \in \mathbb{Z}$. Replace F with $(\_ \otimes L^\vee[-m]) \circ F$. By construction we now have $F(\mathcal{O}_X) \cong \mathcal{O}_Y$. If we prove that ω_Y or $\omega_Y^\vee$ is ample, then we are done by lemma 2.5.

Let $P \in \mathrm{D^b}(Y)$ be a point-like object. By lemma 4.5 there exists a closed point $y \in Y$ and an integer m such that $\mathrm{Hom}(P, K(y)[m]) \neq 0$. On the other hand, there are also closed points $x_P, x_y \in X$ and integers $m_P, m_y \in \mathbb{Z}$ such that $F(K(x_P)[m_P]) \cong P$ and $F(K(x_y)[m_y]) \cong K(y)$, since F is an equivalence. Now we compute

$$0 \neq \mathrm{Hom}(P, K(y)[m]) = \mathrm{Hom}(F(K(x_P)[m_P]), F(K(x_y)[m_y + m])) = \mathrm{Hom}(K(x_P)[m_P], K(x_y)[m_y + m])$$

which implies $x_P = x_y$, and hence

$$P \cong F(K(x_P)[m_P]) \cong F(K(x_y)[m_P]) \cong K(y)[m_P - m_y].$$

In fact, since we have arranged that $F(\mathcal{O}_X) = \mathcal{O}_Y$ we get an induced bijection between the closed points of X and those of Y: $F(K(x)) = K(y)[m]$, for some $y \in Y$ and $m \in \mathbb{Z}$, and

$$0 \neq \mathrm{Hom}(\mathcal{O}_X, K(x)) = \mathrm{Hom}(F(\mathcal{O}_X), F(K(x)) = \mathrm{Hom}(\mathcal{O}_Y, K(y)[m])$$

implies that $m = 0$.

Finally, we will prove that if ω_X^k is very ample, then ω_Y^k is very ample. We follow [H] and show this by proving that ω_Y^k separates points and tangents. The line bundle ω_Y^k separates points, if for any two closed points $y_1, y_2 \in Y$ the restriction map

$$\omega_Y^k \to \omega_Y^k(y_1) \oplus \omega_Y^k(y_2) \cong K(y_1) \oplus K(y_2)$$

induces a surjection $H^0(Y, \omega_Y^k) \to H^0(Y, K(y_1) \oplus K(y_2))$. Note that there is only one map $\omega_Y^k \to K(y_i)$ (up to scaling). Hence we get the desired result by transport of structure, since $F(\omega_X) = \omega_Y$.

Now, on to separating tangents. The line bundel ω_Y^k separates tangents if for every subscheme $Z_y \subset Y$ of length 2 the restriction map $\omega_Y^k \to \mathcal{O}_{Z_y}$ induces a surjection $H^0(Y, \omega_Y^k) \to H^0(Y, \mathcal{O}_{Z_y})$. Let us unwrap this a bit: the subscheme Z_y is given by a point y, and a tangent vector at y; and $\mathcal{O}_{Z_y}$ fits into a short exact sequence

$$0 \to K(y) \to \mathcal{O}_{Z_y} \to K(y) \to 0.$$

Thus, $\mathcal{O}_{Z_y}$ corresponds to a class in $\mathrm{Hom}(K(y), K(y)[1])$, which we may transport to $\mathrm{D}^{\mathrm{b}}(X)$ to obtain a sheaf $\mathcal{O}_{Z_x}$ belonging to some length 2 subscheme $Z_x \subset X$. The equivalence F maps the restriction map $\omega_X^k \to \mathcal{O}_{Z_x}$ to the restriction map $\omega_Y^k \to \mathcal{O}_{Z_y}$. This proves the result. QED

7. Ample sequences

7.1 Definition. Let $\mathcal{A}$ be a k-linear abelian category with finite-dimensional homsets. A sequence of objects $L_i \in \mathcal{A}$, $i \in \mathbb{Z}$, is called *ample* if for any object $A \in \mathcal{A}$ there exists an integer $i_0(A)$ such that for all $i < i_0(A)$ one has:

1. The natural morphism $\mathrm{Hom}(L_i, K) \otimes_k L_i \to A$ is surjective.
2. If $j \neq 0$, then $\mathrm{Hom}(L_i, A[j]) = 0$.
3. $\mathrm{Hom}(A, L_i) = 0$.

7.2 Example. Let X be a smooth projective scheme over K. Let L be an ample line bundle on X. Then the sequence $(L^i)_{i\in\mathbb{Z}}$ forms an ample sequence in the category of coherent sheaves on X. To see this, apply Serre's vanishing theorem, to see that the first two conditions are satisfied. The third condition also follows from Serre's vanishing theorem after applying Serre duality.

7.3 Proposition ([BO]; prp 4.23 of [H]). *Let $F\colon \mathrm{D}^{\mathrm{b}}(\mathcal{A}) \to \mathrm{D}^{\mathrm{b}}(\mathcal{A})$ be an autoequivalence, with $\mathcal{A}$ a k-linear abelian category with finite-dimensional homsets. Suppose $f\colon \mathrm{id}_{\{L_i\}} \to F|_{\{L_i\}}$ is an isomorphism of functors on the full subcategory $\{L_i\}$ given by an ample sequence L_i in $\mathcal{A}$. Then there exists a unique extension to an isomorphism of functors $\tilde{f}\colon \mathrm{id} \to F$.*

8. Autoequivalences

Let X be a smooth projective variety over K. In this section we want to study the group of autoequivalences of $\mathrm{D}^{\mathrm{b}}(X)$.

8.1 Example. We give three examples of autoequivalences. As we will see, if ω_X or $\omega_X^\vee$ is ample, then these examples essentially cover all autoequivalences.

1. Let $f\colon X \to X$ be an automorphism. The $f_*\colon \mathrm{D^b}(X) \to \mathrm{D^b}(X)$ is an equivalence; its inverse is given by f^*.
2. The shift functor [1] is an equivalence.
3. Let L be a line bundle on X. Then $\_ \otimes L$ is an equivalence.

8.2 Proposition ([BO]; prp 4.17 of [H]). *Let X be a smooth projective variety over K. Assume that ω_X or $\omega_X^\vee$ is ample. Then*

$$\mathrm{Aut}(\mathrm{D^b}(X)) = \mathbb{Z} \times (\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X)).$$

Proof. Let F be an autoequivalence. After shifting and twisting by a line bundle we may assume that $F(\mathcal{O}_X) = \mathcal{O}_X$. Since F commutes with the Serre functor S_X, we find $F(\omega_X^k) = \omega_X^k$, for all $k \in \mathbb{Z}$. Thus F induces a graded automorphism of B_X, and thus an automorphism ϕ of $\mathrm{Proj}(B_X) = X$. After replacing F with $F \circ \phi^*$, we may assume that the induced automorphism on X is trivial. Now it follows from proposition 7.3 that F is the identity. QED

9. Bibliography

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