Birational Calabi-Yau varieties have the same Betti numbers

Notes by Johan Commelin Friday, the 9th of March, 2018

1. Introduction

1.1. The goal of this talk is the following theorem of Batyrev.

Let X and Y be two birational Calabi–Yau varieties over $\mathbb{C}$. Then X and Y have the same Betti numbers: $\dim \mathrm{H}^i(X^{\mathrm{an}}, \mathbb{Q}) = \dim \mathrm{H}^i(Y^{\mathrm{an}}, \mathbb{Q})$ *for all* $i \geq 0$. (thm 1.1 of [Bat])

Recall that a connected smooth projective variety $X/\mathbb{C}$ of dimension n is a *Calabi–Yau variety* if the canonical bundle Ω^n_X is trivial.

In what follows we will write down a numbered sequence of statements that start with the assumption of the theorem and end with its conclusion. The rest of the talk will consist in proving that each statement implies the one that follows it.

1.2. Let X and Y be connected smooth projective complex varieties.

(A) *X and Y are birational Calabi–Yau varieties.*

(B) *X and Y are K-equivalent varieties.*

The varieties X and Y are called *K-equivalent* if there exists an auxilliary smooth projective variety Z with proper birational maps $f\colon Z \to X$ and $g\colon Z \to Y$

$$\begin{array}{ccc} & Z & \\ {}^{f}\swarrow & & \searrow^{g} \\ X & & Y \end{array}$$

such that $f^*\Omega^n_X \cong g^*\Omega^n_Y$. (Note that K-equivalent varieties are birational.)

Fix such data Z, f, and g. Observe that there exists a finitely generated integrally closed subdomain $\mathcal{O} \subset \mathbb{C}$ such that X, Y, Z, f, and g are defined over $\mathcal{O}$, and such that $f^*\Omega^n_{X/\mathcal{O}} \cong g^*\Omega^n_{Y/\mathcal{O}}$. From now on we will refer to the data over $\mathbb{C}$ by $X_{\mathbb{C}}$, $Y_{\mathbb{C}}$, $Z_{\mathbb{C}}$, $f_{\mathbb{C}}$, and $g_{\mathbb{C}}$.

(C) *For every maximal ideal $\mathfrak{p} \subset \mathcal{O}$ and every finite extension $\mathbb{F}_{q^r}$ of the finite field $\mathbb{F}_q = \mathcal{O}/\mathfrak{p}$ the $\mathcal{O}$-schemes X and Y have the same number of $\mathbb{F}_{q^r}$-rational points:* $|X(\mathbb{F}_{q^r})| = |Y(\mathbb{F}_{q^r})|$.

Let $F \subset \mathbb{C}$ be the field generated by $\mathcal{O}$, and let $\bar{F}$ be the algebraic closure of F in $\mathbb{C}$.

(D) *There exists a prime number ℓ such that for all $i \in \mathbb{Z}_{\geq 0}$ the* $\mathrm{Gal}(\bar{F}/F)$*-representations on* $\mathrm{H}^i_{\text{ét}}(X_{\bar{F}}, \mathbb{Q}_\ell)$ *and* $\mathrm{H}^i_{\text{ét}}(Y_{\bar{F}}, \mathbb{Q}_\ell)$ *are isomorphic up to semisimplification.*

(Recall that the Tate conjecture implies these Galois representations are semisimple. But as long as this conjecture is open, we will just semisimplify to get an unconditional result.)

(E) *The varieties $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ have the same Hodge numbers:* $\dim \mathrm{H}^i(X_{\mathbb{C}}, \Omega^j_{X_{\mathbb{C}}/\mathbb{C}}) = \dim \mathrm{H}^i(Y_{\mathbb{C}}, \Omega^j_{Y_{\mathbb{C}}/\mathbb{C}})$.

(F) *The varieties $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ have the same Betti numbers:* $\dim \mathrm{H}^i(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}) = \dim \mathrm{H}^i(Y^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q})$.

1.3 Some words on the proof. We follow the proof of [Ito], which mainly focuses on the implications (B) $\implies$ (E). This means that Ito relaxes the assumptions and strengthens the final result. Nevertheless, both Batyrev and Ito follow the same global strategy, and the crucial intermediate step (C) is in both cases attained by methods of p-adic integration. Batyrev then goes directly from (C) to (F) using the Weil conjectures (proved by Deligne) and the Lefschetz trace formula for étale cohomology. Ito refines this with the intermediate steps (D) and (E) by using Chebotarev's density theorem and p-adic Hodge theory respectively.

1.4. We conclude the introduction by remarking that there exist examples of birational Calabi–Yau varieties that are non-isomorphic. In particular, the results of Batyrev and Ito are non-trivial.

2. Straightforward: from (A) to (B) and from (E) to (F)

2.1. We first show (A) $\implies$ (B). Let X and Y be birational complex Calabi–Yau varieties. Let Z be the resolution of singularities of the closure of the graph of a birational map inside $X \times Y$. We automatically get proper birational maps $f\colon Z \to X$ and $g\colon Z \to Y$. Since Ω^n_X and Ω^n_Y are trivial by assumption, we see that $f^*\Omega^n_X \cong \mathcal{O}_Z \cong g^*\Omega^n_Y$. Thus X and Y are K-equivalent.

More generally, if X and Y are birational smooth projective varieties and their canonical bundles are nef, then one can show that X and Y are K-equivalent. (Recall that a line bundle or divisor D on X is called *nef* if for every curve C on X we have $C \cdot D \geq 0$. In particular, the canonical bundle of a Calabi–Yau variety is nef.) See proposition 2.1 of [Ito] for a proof of this claim.

2.2. Now we show (E) $\implies$ (F). For a smooth projective variety X over some field $F \subset \mathbb{C}$, there is the Hodge–de Rham spectral sequence

$$\mathrm{H}^i(X, \Omega^j_{X/F}) \Rightarrow \mathrm{H}^{i+j}_{\mathrm{dR}}(X/F).$$

Note that this is compatible with extension of scalars:

$$\mathrm{H}^i(X, \Omega^j_{X/F}) \otimes_F \mathbb{C} = \mathrm{H}^i(X_{\mathbb{C}}, \Omega^j_{X_{\mathbb{C}}/\mathbb{C}}) \quad \text{and} \quad \mathrm{H}^{i+j}_{\mathrm{dR}}(X/F) \otimes_F \mathbb{C} = \mathrm{H}^{i+j}_{\mathrm{dR}}(X_{\mathbb{C}}/\mathbb{C}).$$

By GAGA, we have a canonical isomorphisms

$$\mathrm{H}^i(X_{\mathbb{C}}, \Omega^j_{X_{\mathbb{C}}/\mathbb{C}}) = \mathrm{H}^i(X^{\mathrm{an}}_{\mathbb{C}}, \Omega^j_{X^{\mathrm{an}}_{\mathbb{C}}}) \quad \text{and} \quad \mathrm{H}^{i+j}_{\mathrm{dR}}(X_{\mathbb{C}}/\mathbb{C}) = \mathrm{H}^{i+j}_{\mathrm{dR}}(X^{\mathrm{an}}_{\mathbb{C}}).$$

For completeness we finally mention de Rham's theorem, $\mathrm{H}^{i+j}_{\mathrm{dR}}(X^{\mathrm{an}}_{\mathbb{C}}) = \mathrm{H}^{i+j}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{C})$, which boils down to the observation that the de Rham complex $\Omega^\bullet_{X^{\mathrm{an}}_{\mathbb{C}}}$ is a resolution of the constant sheaf $\mathbb{C}$. See, e.g., [Gro] for details.

3. From (B) to (C)

3.1. Fix a finitely generated integrally closed subdomain $\mathcal{O} \subset \mathbb{C}$ and a diagram of smooth projective n-dimensional $\mathcal{O}$-schemes

$$\begin{array}{ccc} & Z & \\ {}^{f}\swarrow & & \searrow^{g} \\ X & & Y \end{array}$$

such that f and g are proper birational morphisms and such that $f^*\Omega^n_{X/\mathcal{O}} \cong g^*\Omega^n_{Y/\mathcal{O}}$.

Let F be the field of fractions of $\mathcal{O}$. Note that F is a finitely generated field over $\mathbb{Q}$. For purposes of this talk, we assume that F is a number field, so that $\mathrm{Spec}(\mathcal{O})$ is an open subscheme of $\mathrm{Spec}(\mathcal{O}_F)$, where $\mathcal{O}_F$ is the ring of integers of F. The general case of finitely generated fields can be dealt with by specialisation arguments.

Note that we are in situation (B). Our goal is to prove (C). So fix a maximal ideal $\mathfrak{p} \subset \mathcal{O}$, and let $F_\mathfrak{p}$ be the $\mathfrak{p}$-adic completion of F. Write $\mathcal{O}_\mathfrak{p}$ for the ring of integers of $F_\mathfrak{p}$, and let q be the cardinality of the finite field $\mathbb{F}_q = \mathcal{O}/\mathfrak{p}$.

3.2. In the previous talk we saw that there exists a canonical measure on the set of points $X(\mathcal{O}_\mathfrak{p})$. By the theorem of Weil we know that

$$\int_{X(\mathcal{O}_\mathfrak{p})} \mathrm{d}\mu_{\mathrm{can}} = \frac{|X(\mathbb{F}_q)|}{q^n} \quad \text{and} \quad \int_{Y(\mathcal{O}_\mathfrak{p})} \mathrm{d}\mu_{\mathrm{can}} = \frac{|Y(\mathbb{F}_q)|}{q^n}.$$

Thus it suffices to show that $\int_{X(\mathcal{O}_\mathfrak{p})} \mathrm{d}\mu_{\mathrm{can}} = \int_{Y(\mathcal{O}_\mathfrak{p})} \mathrm{d}\mu_{\mathrm{can}}$ in order to conclude that $|X(\mathbb{F}_q)| = |Y(\mathbb{F}_q)|$.

3.3. The construction of the canonical measure generalises in the following way. Fix a sub-line bundle $\mathcal{L} \subset \Omega^n_X$. Let $U_1 \cup U_2 \cup \ldots \cup U_k$ be a finite cover of X by Zariski open S-subschemes of X such that for each i the line bundle $\mathcal{L}_{|U_i}$ is trivial over U_i. This means that we can pick a gauge form $\omega_i \in \Gamma(U_i, \mathcal{L})$. We get a measure μ_{ω_i} on $U_i(\mathcal{O}_\mathfrak{p})$, and this measure does not depend on the choice of ω_i, since any two gauge forms differ by an invertible function s on U_i: hence $|s(x)|_\mathfrak{p} = 1$ for all $x \in U_i(\mathcal{O}_\mathfrak{p})$. This means that the measures μ_{ω_i} glue together to a global measure $\mu_\mathcal{L}$ on the set $X(\mathcal{O}_\mathfrak{p})$. This measure only depends on the isomorphism class of $\mathcal{L}$.

Observe that if $\mathcal{L} = \Omega^n_X$, then $\mu_\mathcal{L} = \mu_{\mathrm{can}}$.

3.4. Now we are in business. Recall that f and g are proper birational maps. Therefore we can apply a change of variables to obtain

$$\int_{X(\mathcal{O}_\mathfrak{p})} \mathrm{d}\mu_{\mathrm{can}} = \int_{X(\mathcal{O}_\mathfrak{p})} \mathrm{d}\mu_{\Omega^n_X} = \int_{Z(\mathcal{O}_\mathfrak{p})} \mathrm{d}\mu_{f^*\Omega^n_X}$$

and similar for g. By assumption $f^*\Omega^n_X \cong g^*\Omega^n_Y$, and therefore $\mu_{f^*\Omega^n_X} = \mu_{g^*\Omega^n_Y}$ on $Z(\mathcal{O}_\mathfrak{p})$. Hence we find $\int_{X(\mathcal{O}_\mathfrak{p})} \mathrm{d}\mu_{\mathrm{can}} = \int_{Y(\mathcal{O}_\mathfrak{p})} \mathrm{d}\mu_{\mathrm{can}}$ which implies $|X(\mathbb{F}_q)| = |Y(\mathbb{F}_q)|$.

By running the same argument over finite unramified extensions of $F_\mathfrak{p}$ we conclude that $|X(\mathbb{F}_{q^r})| = |Y(\mathbb{F}_{q^r})|$. This finishes the proof of (C).

4. From (C) to (D)

4.1. Let q be the cardinality of the finite field $\mathcal{O}/\mathfrak{p}$. As is common, we assemble the number of points over finite extensions $\mathbb{F}_{q^r}$ of $\mathbb{F}_q = \mathcal{O}/\mathfrak{p}$ into the Hasse–Weil zeta function:

$$Z(X, \mathfrak{p}, t) = \exp\left(\sum_{r=1}^{\infty} \frac{|X(\mathbb{F}_{q^r})|}{r} t^r \right).$$

Let $\mathrm{Fr}_{\mathfrak{p}}\colon X_{\bar{\mathbb{F}}_q} \to X_{\bar{\mathbb{F}}_q}$ be the q-th power Frobenius automorphism. Then the set of fixed points of $\mathrm{Fr}_{\mathfrak{p}}$ in $X(\bar{\mathbb{F}}_q)$ is exactly $X(\mathbb{F}_q)$.

We now make the connection to étale cohomology. Let ℓ be a prime number that does not divide q. Then the Lefschetz trace formula tells us that

$$|X(\mathbb{F}_q)| = \sum_{i=0}^{2n} (-1)^i \mathrm{Tr}(\mathrm{Fr}_{\mathfrak{p}}^*; \mathrm{H}^i_{\text{ét}}(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell)).$$

This means that

$$Z(X, \mathfrak{p}, t) = \frac{P_1(X, \mathfrak{p}, t) \cdots P_{2n-1}(X, \mathfrak{p}, t)}{P_0(X, \mathfrak{p}, t) P_2(X, \mathfrak{p}, t) \cdots P_{2n}(X, \mathfrak{p}, t)}$$

where $P_i(X, \mathfrak{p}, t)$ is the characteristic polynomial $\det(1 - \mathrm{Fr}_{\mathfrak{p}}^* t; \mathrm{H}^i_{\text{ét}}(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell))$.

4.2 Theorem (Deligne). *The roots of $P_i(X, \mathfrak{p}, t)$ (i.e., the eigenvalues of $\mathrm{Fr}_{\mathfrak{p}}^*$ acting on $\mathrm{H}^i_{\text{ét}}(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell)$) are algebraic numbers, and all their conjugates have complex absolute value $q^{i/2}$.*

4.3. We see immediately that our assumption (C) implies that $Z(X, \mathfrak{p}, t) = Z(Y, \mathfrak{p}, t)$; and by Deligne's theorem this means that $P_i(X, \mathfrak{p}, t) = P_i(Y, \mathfrak{p}, t)$ for $i = 0, \ldots, n$. And this equality holds for all maximal ideals $\mathfrak{p} \subset \mathcal{O}$.

We can lift the action of $\mathrm{Fr}_{\mathfrak{p}}^*$ on $\mathrm{H}^i_{\text{ét}}(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ to the action of a *Frobenius element* $\tilde{\mathrm{Fr}}_{\mathfrak{p}}^* \in \mathrm{Gal}(\bar{F}/F)$ on $\mathrm{H}^i_{\text{ét}}(X_{\bar{F}}, \mathbb{Q}_\ell)$. This element is well-defined up to conjugation.

Observe that $\mathrm{Tr}(\tilde{\mathrm{Fr}}_{\mathfrak{p}}^*; \mathrm{H}^i_{\text{ét}}(X_{\bar{F}}, \mathbb{Q}_\ell))$ equals $\mathrm{Tr}(\tilde{\mathrm{Fr}}_{\mathfrak{p}}^*; \mathrm{H}^i_{\text{ét}}(Y_{\bar{F}}, \mathbb{Q}_\ell))$ for all maximal ideals $\mathfrak{p} \subset \mathcal{O}$ with $\ell \notin \mathfrak{p}$. By the Chebotarev density theorem, the set of elements $\{\tilde{\mathrm{Fr}}_{\mathfrak{p}}\}_{\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}[1/\ell])^{\mathrm{cl}}}$ generate the group $\mathrm{Gal}(\bar{F}/F)$. Finally, an application of the Brauer–Nesbitt theorem shows that the $\mathrm{Gal}(\bar{F}/F)$-representations $\mathrm{H}^i_{\text{ét}}(X_{\bar{F}}, \mathbb{Q}_\ell)$ and $\mathrm{H}^i_{\text{ét}}(Y_{\bar{F}}, \mathbb{Q}_\ell)$ have isomorphic semisimplifications. This proves claim (D).

5. From (D) to (E)

5.1. Let p be the residue characteristic of $F_{\mathfrak{p}}$. Let $\bar{F}_{\mathfrak{p}}$ be an algebraic closure of the field $F_{\mathfrak{p}}$. Write Γ for the Galois group $\mathrm{Gal}(\bar{F}_{\mathfrak{p}}/F_{\mathfrak{p}})$. Let $\mathbb{C}_p$ be the completion of $\bar{F}_{\mathfrak{p}}$. Write $\mathbb{C}_p(i)$ for $\mathbb{Q}_p(i) \otimes_{\mathbb{Q}_p} \mathbb{C}_p$, with the diagonal action of Γ.

5.2 Theorem (Tate). *We have $\mathbb{C}_p^\Gamma = F_{\mathfrak{p}}$ and $\mathbb{C}_p(i)^\Gamma = 0$ for $i \neq 0$.*

5.3. Denote with B_{HT} the graded $\mathbb{C}_p$-module $\bigoplus_{i \in \mathbb{Z}} \mathbb{C}_p(i)$ with its action of Γ. If V is a finite-dimensional p-adic representation of Γ, we define $\mathrm{D}_{\mathrm{HT}}(V)$ to be $(V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{HT}})^\Gamma$. The graded $\mathbb{C}_p$-modules structure on B_{HT} induces a graded F-module structure on $\mathrm{D}_{\mathrm{HT}}(V)$. In general, we have

$$\dim_{F_{\mathfrak{p}}} \mathrm{D}_{\mathrm{HT}}(V) \leq \dim_{\mathbb{Q}_p} V.$$

5.4 Definition. We call V a *Hodge–Tate representation* if

$$\dim_{F_{\mathfrak{p}}} \mathrm{D}_{\mathrm{HT}}(V) = \dim_{\mathbb{Q}_p} V.$$

5.5 Theorem (Faltings). *The p-adic étale cohomology group* $\mathrm{H}^k_{\text{ét}}(X_{\bar{F}_\mathfrak{p}}, \mathbb{Q}_p)$ *is a Hodge–Tate representation of* Γ. *Also, there exists a canonical and functorial isomorphism*

$$\bigoplus_{i+j=k} \mathrm{H}^i(X, \Omega^j_X) \otimes_F \mathbb{C}_p(-j) \cong \mathrm{H}^k_{\text{ét}}(X_{\bar{F}_\mathfrak{p}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p$$

of Γ*-representations, where* Γ *acts trivially on* $\mathrm{H}^i(X, \Omega^j_X)$ *and diagonally on the right hand side.*

5.6. Let V be a Hodge–Tate representation of Γ. Then we define

$$h^i(V) = \dim_{F_\mathfrak{p}} (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p(i))^\Gamma.$$

We denote with V^{ss} the semisimplification of V. One can show that $h^i(V) = h^i(W)$ for all $i \in \mathbb{Z}$. In particular, V^{ss} is a Hodge–Tate representation.

5.7 Lemma. *We have*

$$\dim_F \mathrm{H}^i(X, \Omega^j_X) = h^j(\mathrm{H}^k_{\text{ét}}(X_{\bar{F}_\mathfrak{p}}, \mathbb{Q}_p)^{\text{ss}})$$

where $k = i + j$.

Proof. Recall that we have $\mathbb{C}_p^\Gamma = F_\mathfrak{p}$ and $\mathbb{C}_p(i)^\Gamma = 0$ for $i \neq 0$. By the theorem above, we have

$$\bigoplus_{i+j=k} \mathrm{H}^i(X, \Omega^j_X) \otimes_F \mathbb{C}_p \cong \mathrm{H}^k_{\text{ét}}(X_{\bar{F}_\mathfrak{p}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p(j).$$

Thus, if we take Galois invariants at both sides, we find

$$\mathrm{H}^i(X, \Omega^j_X) \cong (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p(j))^\Gamma$$

where $V = \mathrm{H}^k_{\text{ét}}(X_{\bar{F}_\mathfrak{p}}, \mathbb{Q}_p)$. Now recall that $h^j(V) = h^j(V^{\text{ss}})$ to conclude. QED

5.8. With the above results, it is now easy to prove (D) $\implies$ (E). By assumption, there is a prime p such that $\mathrm{H}^i_{\text{ét}}(X_{\bar{F}}, \mathbb{Q}_\ell)$ and $\mathrm{H}^i_{\text{ét}}(Y_{\bar{F}}, \mathbb{Q}_\ell)$ are isomorphic up to semisimplification. Thus we have $\dim_F \mathrm{H}^i(X, \Omega^j_X) = \dim_F \mathrm{H}^i(Y, \Omega^j_Y)$, which means we have proven claim (E).

6. References

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