Seminar Local Fields (Freiburg, 2019)

Prof. Dr. A. Huber   Johan Commelin

Time: Wednesday, 8ct
Room: SR 404, Ernst-Zermelo-Str. 1

The real numbers form a completion of the rational numbers, and other completions are given by the so-called p-adic numbers. These are the first examples of local fields. Local fields are a very important concept in the study of number fields (finite extensions of $\mathbb{Q}$), because they allow us to study problems “locally”. For example, one of the main goals in number theory is to study solutions of polynomial equations over the integers or the rationals. This is a very hard problem, but one can make some progress by studying the solutions locally over the p-adic numbers for every prime p. In this seminar we will follow the book “Local Fields” by Serre, and explore the basic properties of local fields. The goals of this seminar are a proof of the local Kronecker–Weber theorem and the statement of local class field theory. At the end of this seminar, students should be well prepared to study the proof of (local) class field theory, one of the highlights of number theory in the previous century.

References

1. Serre, Jean-Pierre. *Local fields.* Translated from the French by Marvin Jay Greenberg. Graduate Texts in Mathematics, 67. Springer-Verlag, New York-Berlin, 1979. viii+241 pp. ISBN: 0-387-90424-7
2. Neukirch, Jürgen. *Algebraische Zahlentheorie.* Springer-Verlag, Berlin, 1992. xiii+595 pp. ISBN: 3-540-54273-6

Bachelor projects

1. Extensions of local fields
2. Witt vectors
3. The Hasse–Arf theorem
4. Local Kronecker–Weber
5. Local class field theory (ambitious!)

Talk 1 — Discrete valuation rings

Date: 24 April 2019

Speaker: TBA

1. Define discrete valuation rings and discuss their basic properties.
2. Explain examples: $\mathbb{Z}_{(p)} \subset \mathbb{Q}$ and $K[[T]]$.
3. Prove that discrete valuation rings are precisely the Noetherian local rings whose maximal ideal is generated by a non-nilpotent element.
4. Prove that discrete valuation rings are precisely the Noetherian integral domains that are integrally closed and have a unique non-zero prime ideal.

Reference: §I.1 – §I.3 of "Local Fields"; §II.3 of "Algebraische Zahlentheorie".

Talk 2 — Extensions

Date: 8 May 2019
Speaker: TBA

1. Define the notions of *ramification index* and *residue degree*.
2. Introduce the norm and inclusion morphisms, and show how they relate to each other.
3. Describe the basics of ramification for local extensions (unramified, totally ramified, connection with Eisenstein polynomials).
4. Consider Galois extensions, and definethe inertia group.
5. Describe how these groups behave in composite extensions $K \subset L \subset M$.
6. Define Frobenius elements.

Reference: §I.4 – §I.8 of "Local Fields"; §II.7 and §II.9 of "Algebraische Zahlentheorie".

Talk 3 — Topology

Date: 15 May 2019
Speaker: TBA

1. Discuss completion of a field with respect to a discrete absolute value (both metric completion and algebraic completion).
2. Characterise the normalised absolute value (algebraically and analytically).
3. Show that a field extension of a field that is complete with respect to a discrete valuation has a unique extension of that valuation.
4. Discuss the interplay between field extensions and completions.

Reference: §II.1 – §II.3 of "Local Fields"; §II.4 and §II.8 of "Algebraische Zahlentheorie".

Talk 4 — Witt vectors

Date: 22 May 2019

Speaker: TBA

1. Construct the Witt vectors, and describe their ring structure.
2. Discuss "Frobenius and Verschiebung".
3. If time permits, give different perspectives on, and properties of, the Witt vectors.
4. Witt vectors are needed as machinery for the next talk; but they are also interesting in their own right.

Reference: §II.6 of "Local Fields"; p. 139 of "Algebraische Zahlentheorie". Also see the relevant section in "Algebra" by Bosch.

Talk 5 — Structure of complete DVRs

Date: 29 May 2019
Speaker: TBA

1. Show that an equicharacteristic complete discrete valuation ring is isomorphic to the ring of power series over its residue field.
2. Discuss the structure of unequal characteristic complete discrete valuation rings: they are in a unique way a ramified extension of the ring of Witt vectors of their residue field.

Reference: §II.4 – §II.5 of "Local Fields".

Talk 6 — Discriminants and Differents

Date: 5 June 2019
Speaker: TBA

1. Define the *discriminant* of a lattice with respect to a bilinear form.
2. Apply the notion of discriminant to an extension of local fields.
3. Introduce the *different* and show how it relates to the discriminant.
4. Discuss elementary properties of discriminant and different (transitivity, localisation, completion).
5. Make the connection between discriminant and ramification.

Reference: §III.1 – §III.4 of "Local Fields"; §II.4 and §II.8 of "Algebraische Zahlentheorie".

Talk 7 — Higher ramification groups

Date: 19 June 2019
Speaker: TBA

1. Introduce the higher ramification groups.

2. Show how the higher ramification groups behave under field extensions.
3. Introduce the upper-numbering and show how to move back and forth between lower and upper-numbering for ramification groups.
4. Clearly state the Hasse–Arf theorem; it is the motivation for the next two talks.

Reference: §II.10 of "Algebraische Zahlentheorie"; §IV.1 – §IV.4 of "Local Fields".

Talk 8 — The norm map

Date: 26 June 2019
Speaker: TBA

1. Recall the subgroups $U_L^{(i)} \subset L^*$.
2. If L/K is unramified, show that the norm maps $U_L^{(i)}$ into and onto $U_K^{(i)}$.
3. Similarly discuss how the case of a totally ramified cyclic extension of prime order.

Reference: Proposition IV.2.7 and §V.1 – §V.3 of "Local Fields"; §II.10 of "Algebraische Zahlentheorie".

Talk 9 — The Hasse–Arf theorem

Date: 3 July 2019
Speaker: TBA

1. Prove the Hasse–Arf theorem.

Reference: §V.4 – §V.7 of "Local Fields"; §V.6 of "Algebraische Zahlentheorie".

Talk 10 — Kummer theory

Date: 10 July 2019
Speaker: TBA

1. Define Kummer extensions.
2. State and prove Kummer theory.
3. If time permits, state Artin–Schreier theory.

Reference: §IV.3 of "Algebraische Zahlentheorie"; §X.3 of "Local Fields".

Talk 11 — Local Kronecker–Weber

Date: 17 July 2019
Speaker: TBA

1. Prove the local Kronecker–Weber theorem.

Reference: §V.1 of "Algebraische Zahlentheorie".
Rosen, Michael. *An elementary proof of the local Kronecker–Weber theorem.* Trans. Amer. Math. Soc. 265 (1981), no. 2, 599–605.

Talk 12 — Statement of local class field theory

Date: 24 July 2019
Speaker: TBA

1. Introduce the local reciprocity map.
2. State the theorem that finite abelian extensions of a local field K are in bijection with the finite index subgroups of K^*.
3. Illustrate the theorem with the (trivial) example $K = \mathbb{R}$.

Reference: Theorem V.1.4 of "Algebraische Zahlentheorie"; §XIII of "Local Fields".